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The Geometry of the Moduli Space of Torsion Free Sheaves on Projective Spaces

Geometria dos Espaços de Moduli de Feixes Sem Torção em Espaços Projetivos

Campinas

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Resumo

Nosso objetivo neste tabalho é estudar a geometria dos espaços de móduli de feixes de posto 2 no espaço projetivo. Nós apresentamos uma nova família de mônadas cuja cohomologia é um fibrado vetorial de posto 2 estável em $\mathbb{P}^3$. Também estudaremos a irredutibilidade, suavidade e uma descrição geométrica e algumas dessas famílias. Tais fatos são usados para demonstrar que o espaço de móduli de fibrados estáveis de posto 2 em $\mathbb{P}^3$, com primeira classe de Chern igual 0 e segunda classe de Chern igual a 5 tem exatamente 3 componentes irredutíveis. Adicionalmente, descrevemos novas componentes irredutíveis do espaço de móduli de feixes sem torção, semi-estáveis de posto 2 em $\mathbb{P}^3$, cujos pontos genéricos não são localmente livres. Como aplicação, provamos que o número de tais componentes cresce indefinidamente junto com a segunda classe de Chern. Provamos ainda que $\mathcal{M}(-1, 2, 4)$ é irredutível, que $\mathcal{M}(-1, 2, 2)$ possui pelo menos duas componentes irredutíveis e que essas componentes têm interseção não vazia, e que $\mathcal{M}(-1, 2, 0)$ possui pelo menos 4 componentes, onde pelo menos 3 destas possui interseção não vazia.

Palavras-chave: Feixes sem torção. Espaços de moduli. Fibrados vetoriais. Problemas de classificação.

Abstract

Our goal is to study the geometry of moduli spaces of rank 2 sheaves on projective spaces. We present a new family of monads whose cohomology is a stable rank two vector bundle on $\mathbb{P}^3$. We also study the irreducibility and smoothness together with a geometrical description of some of these families. Such facts are used to prove that the moduli space of stable rank two vector bundles of zero first Chern class and second Chern class equal to 5 has exactly three irreducible components. Additionally, we describe new irreducible components of the moduli space of rank 2 semistable torsion free sheaves on the tridimensional projective space whose generic point corresponds to non-locally free sheaves. As applications, we prove that the number of such components grows as the second Chern class grows. Additionally, we proved that $\mathcal{M}(-1,2,4)$ is irreducible; $\mathcal{M}(-1,2,2)$ has at least two irreducible components, such that the intersection is non-empty and that $\mathcal{M}(-1,2,0)$ has at least four irreducible components, and that at least three of them has non-empty intersection.

Keywords: Torsion-free sheaves. Vector Bundles. Moduli Spaces. Classification Problems.

Contents

Introduction

The study of vector bundles is a topic that has attracted the attention of the mathematical community, due to its connections to various fields of the mathematics and physics, see for instance [24].

After the proof of existence of a projective moduli scheme parametrizing S-equivalence classes of semistable sheaves on a projective variety by Maruyama [49], the study of the geometry of such moduli spaces has been a central topic of research within algebraic geometry. Although a lot is known for curves and surfaces, general results for three dimensional varieties are still lacking. In fact, moduli spaces of sheaves on 3-folds turn out to be quite complicated spaces (as it is illustrated by Vakil's Murphy's law [64]), particularly with several irreducible components of various dimensions.

The goal of this thesis is to advance on the study of the moduli space of semistable rank 2 sheaves on $\mathbb{P}^3$ with fixed Chern classes (c_1, c_2, c_3). We will denote by $\mathcal{M}(c_1, c_2, c_3)$ the moduli space of rank 2 torsion free sheaves with Chern classes (c_1, c_2, c_3), and additionally, we will also consider the open subset consisting of stable rank 2 reflexive sheaves, denoted by $\mathcal{R}(c_1, c_2, c_3)$; when $c_3 = 0$, this is actually the moduli space of stable rank 2 locally free sheaves, and that will be denoted by $\mathcal{B}(c_1, c_2)$. Questions on the geometry of such spaces, such as connectedness, or the number of irreducible components, seem to be less explored if compared to the study of the geometry of the Hilbert schemes of curves in the projective 3-space for instance. (Some known results for the Hilbert schemes can be found in [21, 40, 41, 53]).

A rich literature on these moduli spaces was produced, especially between the 1970's and 1990's, studying $\mathcal{R}(c_1, c_2, c_3)$ and $\mathcal{B}(c_1, c_2)$ for specific values of the Chern classes. For instance, the geometry of $\mathcal{B}(0, c_2)$ and $\mathcal{B}(-1, c_2)$ is completely understood for c_2 up to 4, see [5, 12, 12, 18, 23] for $c_1 = 0$, and [4, 28] for $c_1 = -1$. In addition, Ein characterized an infinite series of irreducible components of $\mathcal{B}(c_1, c_2)$ and proved that the number of irreducible components of $\mathcal{B}(c_1, c_2)$ goes to infinity as the c_2 goes to infinity [15].

Regarding reflexive sheaves, $\mathcal{R}(c_1, c_2, c_3)$ is known for $c_1 = -1, 0$, $c_2 \leqslant 3$ and all possible values for c_3, see [11]. Some extremal values are also known, namely, $\mathcal{R}(-1, c_2, c_2^2)$ was studied by Hartshorne in [25], Chang described $\mathcal{R}(0, c_2, c_2^2 - 2c_2 + 4)$ in [13], while Miró-Roig studied $\mathcal{R}(-1; c_2; c_2^2 - 2c_2 + 4)$ in [52], and the moduli spaces $\mathcal{R}(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ for $1 \leqslant r \leqslant (-1 + \sqrt{4c_2 - 7})/2$, and c_2 greater than 5; and $\mathcal{R}(-1, c_2, c_2^2 - 2(r-1)c_2)$ for c_2 greater than 8 in [50].

Even less is known for torsion free sheaves. The most general results are due to

Okonek and Spindler, who proved in [57] that $\mathcal{M}(0, c_2, c_2^2 - c_2 + 2)$ and $\mathcal{M}(-1, c_2, c_2^2)$ are irreducible for $c_2 \geqslant 6$. For small values of c_2, Miró-Roig and Trautmann proved that $\mathcal{M}(0, 2, 4)$ is irreducible, while Le Potier showed in [46, Chapter 7] that $\mathcal{M}(0, 2, 0)$ has exactly 3 irreducible components; more recently, it was shown in [36] that $\mathcal{M}(0, 2, 0)$ is connected. Trautmann has also argued that $\mathcal{M}(0, 2, 2)$ has exactly 2 irreducible components [63]. Very recently, using stability conditions, Schimidt [59] obtained a new proof for the irreducibility of the moduli spaces of sheaves with maximal third Chern class, thus extending Okonek and Splinder's results for the remaining Chern classes.

Due to the ideas of Bridgeland in his seminal works in [8] and [9], the interest in the study of the geometry of moduli spaces of sheaves, even for more general schemes, was renewed, as can be seeing in the papers [6], [54], [59] and in the references therein. But, despite the power of these new techniques, questions about the connectedness, number and examples of irreducible components of moduli space of sheaves seem to be out of the reach for them, at least at the moment. Thus, this justifies the use of the classical methods employed in the works [32, 34, 35, 36, 42, 43, 60] and in this thesis.

Our starting point will be the study of the moduli spaces of locally free sheaves with first Chern class equals to 0. It is more or less clear from the table in [27, Section 5.3] that $\mathcal{B}(0, 1)$ and $\mathcal{B}(0, 2)$ should be irreducible, while $\mathcal{B}(0, 3)$ and $\mathcal{B}(0, 4)$ should have exactly two irreducible components; see [19] and [12], respectively, for the proof of the statements about $\mathcal{B}(0, 3)$ and $\mathcal{B}(0, 4)$. For $n \geqslant 5$, two families of irreducible components have been studied, namely the *instanton components*, whose generic point corresponds to an instanton bundle, that is, locally free sheaves that are cohomology of monads of the form :

$$0 \to k.\mathcal{O}_{\mathbb{P}^3}(-1) \to (2k+2).\mathcal{O}_{\mathbb{P}^3} \to k.\mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

and the *Ein components*, whose generic point corresponds to a bundle given as cohomology of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0,$$

where $b \geqslant a \geqslant 0$ and $c > a + b$. All of the components of $\mathcal{B}(0, n)$ for $n \leqslant 4$ are of either of these types; Here we focus on a new family of bundles that appear as soon as $n \geqslant 5$. More precisely, we study the family of vector bundles in $\mathcal{B}(0, a^2 + k)$ for each $a \geqslant 2$ and $k \geqslant 1$ which arise as cohomologies of monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \to k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0 \qquad (1)$$

which will be denoted by $\mathcal{G}(a,k)$. We provide a bijection between such monads and monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

where $\tilde{E}$ is a rank 4 instanton bundle of charge k. When $k=1$ these facts, are used to prove our first main result.

Main Theorem 1. For each $a \geqslant 2$ not equal to 3, $\mathcal{G}(a,1)$ is a nonsingular open subset of an irreducible component of $\mathcal{B}(0,a^2+1)$ of dimension

$$4 \cdot \binom{a+3}{3} - a - 1.$$

Our second main result provides a complete description of all the irreducible components of $\mathcal{B}(5)$.

Main Theorem 2. The moduli space $\mathcal{B}(0,5)$ has exactly 3 irreducible components, namely:

(i) the *instanton component*, of dimension 37, which consists of those bundles given as cohomology of monads of the form

$$0 \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to 12 \cdot \mathcal{O}_{\mathbb{P}^3} \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \quad \text{and} \tag{2}$$

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \to 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 3 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(2) \to 0; \tag{3}$$

(ii) the *Ein component*, of dimension 40, which consists of those bundles given as cohomology of monads of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0; \tag{4}$$

iii) the closure of the family $\mathcal{G}(2,1)$, of dimension 37, which consists of those bundles given as cohomology of monads of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \quad \text{and} \tag{5}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 6 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{6}$$

Indeed, Hartshorne and Rao proved in [27] that every stable rank 2 bundle on $\mathbb{P}^3$ with Chern classes $c_1(E) = 0$ and $c_1(E) = 5$ is the cohomology of one of the monads listed above. Rao showed in [58] that bundles given as cohomology of monads of the form (3) lie in the closure of the family of instanton bundles of charge 5, which was shown to be irreducible firstly by Coanda, Tikhomirov and Trautmann in [14]; see also [61]. The irreducibility of the family of bundles which arise as cohomology of monads of the form (4) was established by Ein in [15].

Finally, our first main result yields the third component, and we also show that the family of bundles given by the monads of the form (6) lies in the closure of the family $\mathcal{G}(2,1)$.

After this, we proceed to the study of proper torsion free sheaves. The crucial starting point was the identification of three different types of torsion free sheaves made by the authors in [36]; more precisely, we have the following defintion.

Definition 1. Let E be a torsion free sheaf on $\mathbb{P}^3$, and set $Q_E := E^{\vee\vee}/E$, which we assume to be nontrivial; we have the following fundamental sequence

$$0 \to E \to E^{\vee\vee} \to Q_E \to 0 \tag{7}$$

and say that E has

- *0-dimensional singularities* if $\dim Q_E = 0$;
- *1-dimensional singularities* if Q_E has pure dimension 1;
- *mixed singularities* if $\dim Q_E = 1$, but Q_E is not pure.

With this definition in mind, a systematic way of producing examples of irreducible components of $\mathcal{M}(0, c_2, 0)$ whose generic point corresponds to a torsion free sheaf with 0-dimensional and 1-dimensional singularities is given in [36]. Furthermore, Ivanov and Tikhomirov, in [32], constructed irreducible components of $\mathcal{M}(0, 3, 0)$ whose generic point corresponds to a torsion free sheaf with mixed singularities.

Our goal then is to generalize the results presented in [32, 36], and show how to produce irreducible components of $\mathcal{M}(c_1, c_2, c_3)$, for values of c_1, c_2 and c_3 also including cases with $c_1 = -1$ and $c_3 \neq 0$, for sheaves with 0-dimensional, 1-dimensional, and mixed singularities. More precisely, we prove the following two statements.

Main Theorem 3. For each $e \in \{-1, 0\}$, let n and m be positive integers such that $en \equiv m (\mathrm{mod}\ 2)$. Let $\mathcal{R}^*$ be a nonsingular, irreducible component of $\mathcal{R}(e, n, m)$ of expected dimension $8n - 3 + 2e$.

- For each $l \geqslant 1$ there exists an irreducible component $\mathrm{T}(e, n, m, l) \subset \mathcal{M}(e, n, m - 2l)$ of dimension $8n-3+2e+4l$ whose generic point $[E]$ satisfies $[E^{\vee\vee}] \in \mathcal{R}^*$ and $\mathrm{length}(Q_E) = l$.

- For each $r \geqslant 2$ and $s \geqslant 1$ such that $2r + 2s \leqslant m + e + 2$, or $r = 1$ and $s = 0$ when $-e = n = m = 1$, there exists an irreducible component $\mathrm{X}(e, n, m, r, s) \subset \mathcal{M}(e, n+1, m+2+c_1-2r-2s)$ of dimension $8n + 4s + 2r + 2 + e$, whose generic point $[E]$ satisfies $[E^{\vee\vee}] \in \mathcal{R}^*$ and Q_E is supported on a line plus s points.

The case $c_1 = 0$ of the first part of the previous theorem is just [36, Theorem 7]; we prove here the case $c_1 = -1$. The second part is a generalization of [32, Theorem 3], which covers the cases $c_1 = 0$, $n = 2$, $m = 2, 4$.

As as application of our constructions, we provide a description of some of the irreducible components of $\mathcal{M}(-1, 2, c_3)$.

Main Theorem 4. The moduli spaces $\mathcal{M}(-1, 2, c_3)$ are connected and

- $\mathcal{M}(-1, 2, 4)$ is irreducible of dimension 11;
- $\mathcal{M}(-1, 2, 2)$ has at least 2 irreducible components of dimensions 11, and 15. Moreover, the intersection between these two irreducible components are non-empty;
- $\mathcal{M}(-1, 2, 0)$ has at least 4 irreducible components, two of them of dimensions 11, and the other with dimension 15 and 19. Moreover, the intersection between 3 of these components are non-empty;

This work is organized as follows. In Chapter 1, we will introduce the main tools and will fix the notation used troughout this work. We will introduce torsion free, reflexive and locally free sheaves on a scheme, define moduli spaces, Chern classes, and then, we will study monads. Finally, we wil give examples of some known irreducible components of moduli spaces of locally free sheaves, and give the notion of the spectrum of torsion free sheaves, which is a valuable tool that help us to compute the number of irreducible component of the moduli spaces.

In Chapter 2, we define the notion of modified instanton bundles, and their relations with bundles given by cohomology of the monads of the form 1. We then compute the dimension of the Ext groups of the sheaves defined by this monad. Next, we study the geometrical properties of these families of sheaves, in order to obtain the proof for the Main Theorem 1. Using these results, and gathering the results of known components of $\mathcal{B}(0, 5)$ we prove the Main Theorem 2. The results of this chapter can be found in the paper [2], that is a joint work with my advisor Marcos Jardim, Alexander Tikhomirov and Serguey Tikhomirov.

In Chapter 3, we start by building up some basic techniques, and preliminary results. We compute the dimensions of the Ext groups of torsion free sheaves in terms of their Chern classes, and use it to produce the examples of irreducible components of the moduli space of torsion free sheaves, and to prove Main Theorem 3. These results are then explored in order to prove that the number of irreducible components of $\mathcal{M}(c_1, c_2, 0)$ whose generic point correspond to a sheaf with 0-, 1-, or mixed dimensional singularities goes to infinity as c_2 goes to infinity, thus showing that the problem of computing the number of irreducible components of $\mathcal{M}(c_1, c_2, 0)$ becomes more complicated for higher valuers of c_2. Then, we compute the exact number of irreducible components of $\mathcal{M}(-1, 2, c_3)$. Finally, we establish the connectedness of $\mathcal{M}(-1, 2, c_3)$, thus completing the proof of Main Theorem 4. The results of this chapter can be found in the paper [1], that is a joint work with my advisor Marcos Jardim and Alexander Tikhomirov.

1 Preliminaries

In this chapter we will introduce the main tools and will fix the notation used throughout this work.In Section 1.1 we will introduce torsion free, reflexive and locally free sheaves on a scheme, which are the main objects that we are interested in this work. In section 1.2 we will make the precise definition of classification of sheaves, by defining moduli spaces, and stating some properties. In Section 1.3 it will be defined Chern classes, which are the main invariants of our study, and in section 1.4, we will study monads, which are one of the main tools to study locally free sheaves on projective spaces. In section 1.5 we will give examples of some known irreducible components of moduli spaces of locally free sheaves, and finally in section 1.6 we will give the notion of spectrum of torsion free sheaves, which are a valuable tool that help us compute the number of irreducible component of the moduli spaces.

1.1 Sheaves of $\mathcal{O}_X$-modules

In this section we will present the main definitions and results about sheaves on schemes that will be used in the next chapters. For more details the reader can see [22, Chapter 2].

Definition 2. Let X be a scheme with a structural sheaf $\mathcal{O}_X$, a **sheaf of $\mathcal{O}_X$-modules** $\mathcal{F}$ over X is a sheaf that satisfies the following conditions:

1. For any open set $U \subset X$ the group $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module.

2. For any open sets $U, V \subset X$ with $V \subset U$ the restriction homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structure via the homomorphism $\rho'_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. That is, if $r \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$, then

$$\rho_{UV}(rm) = \rho'_{UV}(r)\rho_{UV}(m).$$

Definition 3. Let A be a commutative ring with unity, and M a A-module. **The sheaf associated with M over** $Spec(A)$, denoted by $\widetilde{M}$ is defined as follows:

For any prime ideal $p \in A$, let M_p be the localization of M at p. Let $U \subset Spec(A)$ be an open subset, then we define $\tilde{M}(U)$ as the set of all maps:

$$s : U \to \bigsqcup_{p \in U} M_p$$

such that $\forall p \in U$, $s(p) \in M_p$ and s is locally a fraction. More precisely, for any $p \in U$ there is a open neighbourhood of p, V that is contained in U and elements $m \in M$ and $f \in A$ such that for each $q \in V$, $f \notin q$ and $s(q) = m/f$ in M_q. It is possible to prove that $\widetilde{M}$ is a sheaf using the usual restriction maps.

Let $\mathcal{F}$ be a sheaf over a scheme X with structural sheaf $\mathcal{O}_X$. Let $x \in X$, we will denote the **stalk** of $\mathcal{F}$ at x by $\mathcal{F}_x$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X-$modules, then we will denote by $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ the **group of all morphisms from $\mathcal{F}$ to $\mathcal{G}$**, and by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ the **sheaf of morphisms from from $\mathcal{F}$ to $\mathcal{G}$** (sometimes, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is called local hom).

Definition 4. We will say that the sheaf $\mathcal{F}^\vee := \mathcal{H}om(F, \mathcal{O}_\mathcal{X})$ is the dual of $\mathcal{F}$.

Definition 5. Let X be a scheme with strcutural sheaf $\mathcal{O}_X$. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is **quasi-coherent** if X can be covered by affine open sets $U_i = Spec(A_i)$ such that for every i there exists M_i a A_i-module such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. We will say that $\mathcal{F}$ is **coherent** if all M_i are finitely generated A_i-modules.

Definition 6. A coherent sheaf $\mathcal{F}$ over an scheme X is **torsion free**, if for every $x \in X$, the stalk F_x is torsion free $\mathcal{O}_{X,x}-$module, where $\mathcal{O}_X$ is the structural sheaf of X.

It is not hard to see that a coherent sheaf $\mathcal{F}$ over a scheme X is torsion free if, and only if, the canonical map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is injective.

Definition 7. A coherent sheaf $\mathcal{F}$ over an scheme X is **reflexive** if the canonical morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

Note that to say that a sheaf is reflexive is not equivalent to say simply that F is isomorphic to $\mathcal{F}^{\vee\vee}$ (We are asking more, we require the canonical map to be an isomorphism). For an example of sheaf that is isomorphic to its double dual but it is not reflexive the reader can see [17, Corollary 4.15]

Definition 8. Let X be a scheme, with structural sheaf $\mathcal{O}_X$. A sheaf $\mathcal{F}$ of $\mathcal{O}_X-$modules is **locally free**, if, and only if, there is an open cover $\cup_i U_i$ of X, such that $F|_{U_i} \simeq \oplus_{i=1}^{r_i} \mathcal{O}_{U_i}$.

In the above definition, if X is connected, it is possible to prove that $r_i = r$ for every i. In this case, we will say that r is the **rank** of $\mathcal{F}$, and denoted by $\mathrm{rk}\, \mathcal{F}$.

Definition 9. Let X be an algebraic variety. A **vector bundle** E on X is an algebraic variety E, together with a morphism $\pi : E \to X$ satisfying the following conditions:

a) There exists an open cover $\bigcup_{i \in I} U_i$ and a $\mathcal{K}$-vector space V, such that $\psi_i : \pi^{-1}(U_i) \simeq U_i \times V$ is an isomorphism, and $\pi \circ \psi_i^{-1}$ is the projection in the first coordinate.

b) For each $i, j \in I$ the morphism

$$\varphi_{ij} = \psi_j \circ \psi^{-1}_{i|U_i \cap U_j} : (U_i \cap U_j) \times V \to (U_i \cap U_j) \to (U_i \cap U_j) \times V$$

is a linear isomorphism, that is, there exists an invertible matrix $A_{i,j}$, such that

$$\varphi_{ij}(x, v) = (x, A_{ij}(x)v).$$

The matrices $A_{i,j}$ are called **transiction matrices**, if $\dim V > 1$ and **transictions function** if $\dim V = 1$.

Definition 10. Let $\mathcal{F}$ be a vector bundle on a scheme X. If $\operatorname{rk} \mathcal{F} = 1$ we will say that $\mathcal{F}$ is a **line bundle**.

In this work we will not make any distinction between locally free sheaves and vector bundles, and this can be done thanks to the following theorem.

Theorem 11 ([22] - Chapter 2 - Exercise 5.18)**.** There exists an equivalence of the category of locally free sheaves of rank r and the category of vector bundles of rank r over a scheme X.

It is possible to prove that any locally free sheaf is reflexive, and that any reflexive sheaf is torsion free. But the converse is not always true, indeed, it will depend on the dimension of the scheme where the sheaves are defined.

Definition 12. For a fixed sheaf $\mathcal{F}$ on a scheme X, we will denote by $\mathrm{H}^i(X, -)$ the right derived of the functor of global sections of $\mathcal{F}$, $\mathrm{Ext}^i(\mathcal{F}, -)$ the right derived functor of $\mathrm{Hom}(\mathcal{F}, -)$ and by $\mathcal{E}xt^i(\mathcal{F}, -)$ the right derived functor of $\mathcal{H}om(\mathcal{F}, -)$.

The following properties of the sheaves $\mathcal{E}xt^i$ and of the groups Ext^i will be used several times throughout this work.

Proposition 13. Let X be a scheme with structure sheaf $\mathcal{O}_X$, and dualizing sheaf ω_X, L, F and G sheaves of $\mathcal{O}_X-$modules, with L being locally free. The following claims are true:

a) $\mathcal{H}om(\mathcal{O}_X, F) \simeq F$;

b) $\mathcal{E}xt^i(\mathcal{O}_X, F) = 0$ for $i \leqslant 1$;

c) $\mathrm{Ext}^i(\mathcal{O}_X, F) \simeq \mathrm{H}^i(X, F)$;

d) $\mathrm{Ext}^i(F \otimes L, G) \simeq \mathrm{Ext}^i(F, G \otimes L^\vee)$;

e) $\mathcal{E}xt^i(F \otimes L, G) \simeq \mathcal{E}xt^i(F, G \otimes L^\vee) \simeq \mathcal{E}xt^i(F, G) \otimes L^\vee$.

Proof. The items $a) - c)$ follows from [22, Chapter 3, Prop. 6.3], the items $d) - e)$ are proved in [22, Chapter 3, Prop. 6.7]. □

From now on, let X be a noetherian scheme over an algebraically closed field $\mathbb{K}$, and let $\mathrm{Coh}(X)$ denote the category of coherent sheaves on X.

Definition 14. Let $\mathcal{F} \in \mathrm{Coh}(X)$, we have the following definitions:

a) The **support** of $\mathcal{F}$ is the closed set:

$$\mathrm{Supp}(\mathcal{F}) := \{x \in X;\ \mathcal{F}_x \neq 0\};$$

b) The dimension of $\mathrm{Supp}(\mathcal{F})$ is called the **dimension** of $\mathcal{F}$;

c) The **singularity set** of $\mathcal{F}$ is defined as:

$$\mathrm{Sing}(\mathcal{F}) := \{x \in X;\ \mathcal{F}_x \text{ is not locally free over } \mathcal{O}_{X,x}\}.$$

The following propositions will be used several times in this work.

Proposition 15. Let X be a smooth algebraic variety of dimension n, ω_X its dualizing sheaf, and $\mathcal{F} \in \mathrm{Coh}(X)$, of dimension d. Then the sheaves $\mathcal{E}xt^i(\mathcal{F}, \omega_X) = 0$ for all $i < n - d$, and $\mathrm{codim}(\mathcal{E}xt^i(\mathcal{F}, \omega_X)) \geqslant i$, for all $i \geqslant n - d$.

Proof. See [31, 1.1.6] □

Proposition 16. Let X be a smooth algebraic variety of dimension n and $\mathcal{F} \in \mathrm{Coh}(X)$. The following claims are true.

a) $\mathrm{codim}(\mathrm{Sing}(\mathcal{F})) \geqslant 1$.

b) If $\mathcal{F}$ is torsion free then $\mathrm{codim}(\mathrm{Sing}(\mathcal{F})) \geqslant 2$.

c) If $\mathcal{F}$ is reflexive then $\mathrm{codim}(\mathrm{Sing}(\mathcal{F})) \geqslant 3$.

d) If $\mathcal{F}$ is locally free then $\mathrm{Sing}(\mathcal{F}) = \varnothing$.

Proof. See [55, Lemma 1.1.7, Lemma 1.1.8 and Lemma 1.1.9] □

Definition 17. Let X be a smooth algebraic variety of dimension n and $\mathcal{F} \in \mathrm{Coh}(X)$, by definition, $\mathcal{F}|_{X \setminus \mathrm{Sing}(\mathcal{F})}$ is locally free, and we define the **rank** of $\mathcal{F}$ to be the rank of $\mathcal{F}|_{X \setminus \mathrm{Sing}(\mathcal{F})}$.

1.2 Moduli Spaces

The main goal of this work will be to classify the objects that we have defined in the previous section. In this one we will say what to classify formally means. We will need the notion of moduli problem. A moduli problem consist of the following data:

a) A notion of object and equivalence of objects;

b) A notion of family over a base scheme B and equivalence of families;

c) A notion of pullback of families compatible with equivalence.

These notions can be defined at our needs, but they always define a contravariant functor $\mathcal{M} : \mathcal{C} \to \mathrm{Set}$ from the category of the objects that we want to classify to the category of sets, given by

$$\mathcal{M}(X) := \{\text{families over } X\}/ \sim_S$$

$$\mathcal{M}(f : Y \to X) := f^* : \mathcal{M}(X) \to \mathcal{M}(Y).$$

This contravariant functor is called **moduli functor**. We will refer to a moduli problem simply by its moduli functor.

Definition 18. Given a moduli functor $\mathcal{F}$, a scheme M is called **fine moduli space** for the moduli functor $\mathcal{F}$, if M represents $\mathcal{F}$, that is, if $\mathcal{F}$ is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-, M)$.

Example 1. Fix a vector space V over a field k of dimension $n+1$. The projective space $\mathbb{P}^n$ can be interpreted as the fine moduli space of the problem of classifying lines trough the origin in V. For a carefully definition of the notion of this moduli functor the reader can see [30, Example 2.19].

Example 2. Fix a vector space V over a field k of dimension $n + 1$. The moduli problem of classifying all subspaces of fixed dimension d of V has a fine moduli space, called the Grassmannian and denoted by $G(n, d)$. For more details the reader can see [30, Example 2.20].

Unfortunatelly, not all moduli problems can be solved, that is, it is possible to find moduli functor that can not be representable. Even worse, very natural moduli problems cannot be solved, for instance the problem of classifying all endomorphisms of a given vector space does not admmit a fine moduli space (see [30, Example 2.21]), therefore we will need a wider notion for moduli spaces.

Definition 19. We say that a moduli functor $\mathcal{F} : \mathcal{C} \to \mathrm{Set}$ is corepresented by an object $M \in Ob(\mathcal{C})$ if there is a natural transformation $\eta : \mathcal{F} \to \mathrm{Hom}_{\mathcal{C}}(-, M)$ such that $\eta(\{pt\})$ is bijective and for any object $N \in Ob(\mathcal{C})$ and for any natural transformation $\beta : \mathcal{F} \to \mathrm{Hom}_{\mathcal{C}}(-, N)$ there exists a unique morphism $\gamma : M \to N$ such that $\beta = h_\gamma \circ \eta$. The object M is called a **coarse moduli space** for the contravariant moduli functor $\mathcal{F}$.

When a coarse moduli space exists it is not hard to prove that it is unique up to isomorphism, but unfortunately, even coarse moduli spaces can not exist, for instance, the problem of the classification of rank 2 vector bundles of degree 0 on $\mathbb{P}^1$ does not have a coarse moduli space see [30, Example 2.22]. Hence the problem of the classification of rank 2 torsion free sheaves on $\mathbb{P}^3$ which is the goal of this work seems to be out of reach. Then, in order to progress towards the solution of this problem, we need to consider a convenient class of torsion free sheaves, and a convenient notion of equivalence. We will dedicate the rest of this section to this end.

1.2.1 The objects

In this subsection we will define the notion of μ-semistability and Gieseker semistability of sheaves, that are the properties of the class of sheaves which will allow us to actually find a moduli functor and a coarse moduli space. Let E be a torsion free sheaf on a projective scheme over an algebraically closed field k, recall that the Euler characteristic of the sheaf E, denoted by $\chi(E)$ is given by $\sum_{i=1}^{\dim X} (-1)^i \dim \mathrm{H}^i(X, E)$.

Definition 20. Fixing an ample line bundle $\mathcal{O}_X(1)$ on X then the function $P_E(m) = \chi(E \otimes \mathcal{O}_X(m)) = \chi(E(m))$ is **called Hilbert polynomial** of E.

Lemma 21. With the notation of the Definition 20, P_E is a polynomial. Moreover,

$$p_E(m) = \frac{P_E(m)}{\mathrm{rk}\, E}$$

is called the **reduced Hilbert Polynomial of E**

Proof. See [31, Lemma 1.2.1]. □

Given two polynomials, f and g, we will say that $f > g$ or $f \geqslant g$ if for every $m >> 0$, $f(m) > g(m)$ or $f(m) \geqslant g(m)$.

Definition 22. A torsion free sheaf E is **Gieseker semistable** if for any proper subsheaf $F \subset E$ one has $p_F \leqslant p_E$. E is called stable if E the inequality is strict.

For any torsion free sheaf E on a projective variety, X with a fixed ample line bundle $\mathcal{O}_X(1)$, denote by $\alpha_i(E)$ the i-th coefficient of the Hilbert Polynomial of E. Moreover, we will call the number $\alpha_{\dim X-1}(E) - \operatorname{rk} E.\alpha_{\dim X-1}(\mathcal{O}_X)$ **the degree** of the sheaf E and denote it by $\deg E$.

Definition 23. Let E be a torsion free sheaf on a projective variety X, with a fixed line bundle $\mathcal{O}_X(1)$. We define **the slope of** E as the number :

$$\mu(E) := \frac{\deg E}{\operatorname{rk} E}$$

Additionaly, we say that E is $\mu-$**semistable** if, for all subsheaves $F \subset E$ with $0 \leqslant \operatorname{rk} F \leqslant \operatorname{rk} E$, we have $\mu(F) \leqslant \mu(E)$. E is stable if the last inequality is strict.

Remark 1. Clearly the notion of stability depends on the choice of the ample line bundle on the projective variety, we will see the end of this section how to deal with this problem.

Although the moduli space for torsion free sheaves is constructed with the notion of Gieseker semistability, sometimes the $\mu-$semistability is more tractable, and there are spaces where both notions are equivalent as we shall see in the next results.

Proposition 24. Let E be a torsion free sheaf on a projective variety X with fixed ample line bundle $\mathcal{O}_X(1)$. Then we have the following chain of implications.

Proof. See [31, Lemma 1.2.13] □

E is $\mu-$stable $\Rightarrow$ E is Gieseker stable $\Rightarrow$ E is Gieseker semistable $\Rightarrow$ E is $\mu-$semistable.

Proposition 25. For rank 2 locally free sheaf E on $\mathbb{P}^n$, we have that E is Gieseker stability $\Leftrightarrow$ E is $\mu-$stability.

Example 3. The cotangent bundle $\Omega_{\mathbb{P}^n}$ given by the Euler sequence:

$$0 \to \Omega_{\mathbb{P}^n}(1) \to (n+1).\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$$

is Gieseker stable.

Proof. See [31, Lemma 1.4.5]. □

Lemma 26. Let E, E_1 and E_2 be torsion free sheaves on $\mathbb{P}^n$. Then the following claims are true:

a) The sum $E_1 \oplus E_2$ is semistable if, and only if, $\mu(E_1) = \mu(E_2)$.

b) E is $\mu-$(semi)stable, if and only if, $E^\vee$ is.

c) E is $\mu-$(semi)stable, if and only if, $E(k)$ is for $k \in \mathbb{Z}$.

d) If $\deg E$ and rk E are coprime, then E is $\mu-$semistable if, and only if E is $\mu-$stable

Proof. See [55, Lemma 1.2.4], and [51, Page 6]. □

An important tool for checking the μ-stability of vector bundles is the Hoppe's criteria.

Theorem 27 (Hoppe's criteria)**.** Let E be a rank r locally-free sheaf on a smooth projective variety X with cyclic Picard group, then it follows that

a) If $\mathrm{H}^0(X, (\bigwedge^q F)_{\mathrm{norm}}) = 0$ for $1 \leqslant q \leqslant r-1$ then F is $\mu-$stable.

b) If $\mathrm{H}^0(X, (\bigwedge^q F)_{\mathrm{norm}}(-1)) = 0$ for $1 \leqslant q \leqslant r-1$ then F is $\mu-$semistable.

Proof. See [51, Proposition 2.12]. □

1.2.2 The Families

In this subsection we will define the notion of family of sheaves that we are going to classify. Let $f : X \to S$ be a morphism of finite type of Noetherian schemes, for $s \in S$ the fibre $f^{-1}(s) = \mathrm{Spec}(k(s)) \times_S X$ is denoted X_s. Let E be a coherent sheaf on $X \times S$ we will denote E_s the sheaf E restricted to X_s. Often, we will think of E as a collection of sheaves E_s parametrized by $s \in S$.

Definition 28. A flat family of coherent sheaves on the fibres of f is a coherent $\mathcal{O}_X$-module F which is flat over S.

1.2.3 The Moduli Functor

Let X be a projective scheme over an algebraically closed field k of characteristic zero. For a fixed polynomial $P \in \mathbb{Q}[z]$, define the contravariant functor $\mathcal{M}'(P) : (\mathrm{Sch})^o \to \mathrm{Sets}$, where $(\mathrm{Sch})^o$ denote the opposite category of Schemes over $\mathbb{K}$. If $S \in \mathrm{Obj}((\mathrm{Sch})^o)$, $\mathcal{M}'(S)$ is the set of isomorphisms classes of S-flat families of semistable sheaves on X with Hilbert polynomial P. Given a morphism $f : S' \to S$ let $\mathcal{M}'(P)(f)$ be the map obtained by pulling-back sheaves via $f_X = f \times id_X$.

Note that if $F \in \mathcal{M}'(S)$ is a S-flat family of semistable sheaves, and if L is an arbitrary line bundle on S, then $F \otimes p^* L$ is also a S-flat family and all fibers F_s and $F \otimes p^* L$ are isomorphic for every $s \in S$, where $p : L \to S$ is the map of the definition of line bundle. Then, in order to classify families of sheaves on X, it is reasonable to consider the quotient functor $\mathcal{M}(P) \simeq \mathcal{M}'(P)/\sim$, when we say that two families (F_1 and F_2) of sheaves parametrized by S are quivalent, $F_1 \sim F_2$ if, and only if, there is line bundle L on S such that $F_1 \simeq F_2 \otimes p^*L$.

Definition 29. A scheme $M(P)$ is called a moduli space of semistable sheaves if it is a coarse moduli scheme for $\mathcal{M}(P)$.

Definition 30. Let E be a semistable torsion free sheaf. A Jordan-Holder filtration of E is a filtration

$$0 = E_0 \subset E_1 \subset \ldots E_l = E$$

such that the factors $gr_i(E) = E_i/E_{i-1}$ are stable with the same reduzed Hilbert Polynomial $p(E)$.

Proposition 31. Jordan-Holder filtrations always exist. Up to isomorphism, the sheaf $gr(E) = \oplus_i gr_i(E)$ does not depend on the choice of the Jordan-Holder filtration.

Proof. See [31, Chapter 1, Section 1.5] □

Definition 32. Two semistable sheaves E_1 and E_2 with the same reduced Hilbert polynomial are called S-equivalent if $gr(E_1) \simeq gr(E_2)$.

These definitions are important due to the following result.

Lemma 33. Suppose $M(P)$ is the moduli space for $\mathcal{M}(P)$. Then S-equivalent sheaves correspond to identical closed points in M. In particular, if there is a properly semistable sheaf F, (i.e. semistable but not stable), then $\mathcal{M}$ cannot be represented.

Proof. See [31, Lemma 4.1.2] □

We are now in position to state the main result of this section.

Theorem 34. Let X be a projective scheme, and $P \in \mathbb{Q}(z)$. Then there is a scheme $M(P)$ that is a moduli space for $\mathcal{M}(P)$.

Proof. See [31, Chapter 4], or [48]. □

Once we have ensured the existence of the moduli space, we are now allowed to study its geometry. The following result will help us to compute the dimension of the tangent space of the moduli space $M(P)$. Abusing the notation, we often will refer to a sheaf E over X as a point in the moduli space $M(P)$, and denote $E \in M(P)$.

Theorem 35. Let X be a projective scheme, and $P \in \mathbb{Q}(z)$. Then for every $E \in M(P)$ there is a natural bijection between $\mathrm{Ext}^1(E, E)$ and $T_E M(P)$ the tangent space at E in $M(P)$.

Proof. See [26, Theorem 2.6]. □

Two important tools used to compute the Ext of sheaves are the local to global spectral sequence for sheaves and the Serre's duality Theorem.

Theorem 36 (Local-to-global spectral sequence)**.** Let E and F be two sheaves on a projective scheme X. There is a spectral sequence E_r^{pq} with E_1-term

$$E_1^{pq} = \mathrm{H}^q(X, \mathcal{E}xt^p(E, F))$$

which converges to $\mathrm{Ext}^{p+q}(E, F)$.

Proof. See [65, Exercise 30.2K] □

Theorem 37 (Serre Duality)**.** Let X be a projective scheme of dimension n over an algebraically closed field k and let F be a coherent sheaf over X. Let ω_X be the dualizing sheaf on X, and let $\mathcal{O}_X(1)$ be a very ample line bundle on X, then for all $i \geqslant 0$ there is functorial isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \to \mathrm{H}^{n-i}(X, \mathcal{F})^\vee.$$

Proof. See [22, Chapter 3, Thm 7.6]. □

We also will use the following form of Serre Grothendieck duality for coherent sheaves:

Theorem 38. Let X be a smooth projective variety, with dualizing sheaf ω_X, $\mathcal{F}$ and $\mathcal{G}$ two coherent sheaves on X then it holds:

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^{n-i}(\mathcal{G}, \mathcal{F} \otimes \omega_X)^\vee$$

Proof. See [22, Chapter 2, Theorem 7.6]. □

1.3 Chern Classes

In the previous section we established the existence of the moduli space of torsion free sheaves on projective schemes, with fixed Hilbert Polynomial. In this section we are going to introduce the notion of Chern classes, because sometimes, is more convenient to consider them as invariants for the sheaves instead of the Hilbert Polynomial. We will end this section with the Hirzebruch-Riemann-Roch Theorem that show how these two notions are related.

For a quick view on Chern classes the reader can see [22, Appendix A], for a deepder presentation see [16].

Definition 39. Let X be a projective variety over $\mathbb{K}$. A **cycle of codimension r on X** is an element of the free abelian group $Z(X)$ generated by the closed irreducible subvarieties of X of codimension r.

Definition 40. Let $\mathrm{Rat}(X) \subset Z(X)$ be the subgroup generated by differences of the form:

$$\{< \Phi \cap (\{t_0\} \times X) >\} - \{< \Phi \cap (\{t_1\} \times X) >\}$$

where $t_0, t_1 \in \mathbb{P}^1$ and $\Phi \subset \mathbb{P}^1 \times X$ is a subvariety of $\mathbb{P}^1 \times X$ not contained in any fiber $(\{t\} \times X)$.

Definition 41. Two cycles $C_1, C_2 \in Z(X)$ are **rationally equivalent** if there exists if there is a rationally parametrized family of cycles interpolating between them. More precisely, C_1 and C_2 are rationally equivalent if their difference lies in $\mathrm{Rat}(X)$.

Definition 42. For each integer r let $A^r(X)$ be the group of cycles of codimension r on X modulo rational equivalence. The **Chow group** $A(X) = \bigoplus_i A^i(X)$ is the group of cycles modulo rational equivalence. If $Z \in Z(X)$ then $[Z]$ denotes its image in $A(X)$.

Definition 43. Two subschemes $Y_0, Y_1 \subset X$ of a scheme X **intersects transversely** at $p \in Y_0 \cap Y_1$ if p is smooth in Y_0, Y_1 and X, and it holds:

$$T_pY_0 + T_pY_1 = T_pX.$$

Two cycles C_0 and C_1 intersects trasnversely if each irreducible summando of C_0 intersects the irreducible summands of C_1 as schemes. Moreover, Y_0 and Y_1 are **generically transverse** if the Y_0 and Y_1 intersects transversely in the generic point.

Theorem 44. If X is a smooth quasi-projective variety, then there is a unique product structure on $A(X)$ such that for each two generically transverse varieties of X, Y_0 and Y_1, we have that

$$[Y_0] * [Y_1] = [Y_0 \cap Y_1].$$

Proof. See [16, Theorem-Definition 1.5] □

This structures makes $A(X)$ into an associative, commutative ring, graded by codimension, called the **Chow ring**.

It is important to highlight that to define the intersection of cycles, C_0 and C_1, of a smooth quasi-projective variety, X, one can not simply make $[C_0] * [C_1] = [C_0 \cap C_1] \in A(X)$. See [16, Section 1.3.7] for a detailed discussion.

We are now in position to state the main result of this subsection.

Theorem 45. There is a unique way of assigning to each rank r coherent sheaf $\mathcal{F}$, on a smooth quasi-projective variety X a class $c(\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \ldots + c_{\text{top}}(\mathcal{F}) \in A(X)$ where $\text{top} = \min\{r, \ \dim X\}$ that satisfies:

(C1) [Line bundles] Let $\mathcal{F} \cong \mathcal{O}_X(\mathcal{D})$ be the line bundle associated to a divisor $\mathcal{D}$. Then $c(\mathcal{E}) = 1 + \mathcal{D}$.

(C2) [Functoriality] If $f : X' \to X$ is a morphism, then for each i

$$c_i(f^*\mathcal{F}) = f^* c_i(\mathcal{F}),$$

where f^* is the pullback of f.

(C3) [Whitney's formula] If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves on X, then

$$c(\mathcal{F}) = c(\mathcal{F}') \cdot c(\mathcal{F}'').$$

Additionally, for a rank r locally free sheaf $\mathcal{F}$ on a scheme X, we can define the **Chern Polynomial** of $\mathcal{F}$ to be $c(\mathcal{F}) = 1 + c_1(\mathcal{F})t + c_2(\mathcal{F})t^2 + \ldots + c_{\text{top}}(\mathcal{F})t^{\text{top}}$.

Proof. If $\mathcal{F}$ is locally free sheaf the proof can be found in [16, Section 1.3.7], for coherent sheaves the proof is in [47]. $\square$

Finally, we will relate the Chern classes of a torsion free sheaf with its Euler characteristic, thus with its Hilbert Polynomial.

Let $\mathcal{F}$ be a coherent sheaf, with Chern polynomial given by

$$c_t(\mathcal{F}) = \prod_{i=1}^{r}(1 + a_i t)$$

where the a_i are formal symbols obtained by factorizing this polynomial. Then we define the **exponential Chern character**

$$\text{ch}(\mathcal{F}) = \sum_{i=1}^{r} e^{a_i},$$

and the **Todd class** of $\mathcal{F}$,

$$\text{td}(\mathcal{F}) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}},$$

where $\dfrac{x}{1 - e^{-x}} = 1 + \dfrac{1}{2}x + \dfrac{1}{12}x^2 - \dfrac{1}{720}x^4 + \cdots$.

Theorem 46 (Hirzebruch-Riemann-Roch)**.** For a coherent sheaf $\mathcal{F}$ of rank r on a nonsingular projective variety X of dimension n,

$$\chi(\mathcal{F}) = \deg(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(\mathcal{T}_X))_n, \tag{1.1}$$

where $()_n$ denotes the component of degree $n \in A(X) \otimes \mathbb{Q}$.

Proof. A proof that works for coherent sheaves and not only locally free sheaves is less standard, and can be found in [47]. □

Example 4. Given a torsion free sheaf $\mathcal{F}$ on a projective variety, using Hirzebruch-Riemann-Roch, it is possible to prove the $\deg \mathcal{F} = c_1(\mathcal{F})$.

Since the right-hand side of equation 1.1 depend only of the Chern classes of the sheaf, we have that the Euler characteristic, thus the Hilbert polynomial of the sheaf depends only on the Chern classes, and the definitions of the moduli spaces that we have made in the previous sections can be done considering Chern classes instead of Hilbert Polynomials.

For this reason, from now on, we will denote the moduli space of rank r semistable torsion free sheaves with fixed Chern classes, $c_1, ..., c_{\dim X}$ on a smooth projective scheme X by $\mathcal{M}_X(r; c_1, c_2, ..., c_n)$, the open subset of the reflexive sheaves by $\mathcal{R}_X(r; c_1, c_2, ..., c_n)$, and the open subscheme of the locally free sheaves by $\mathcal{B}_X(r; c_1, c_2, ..., c_{\mathrm{top}})$ where $\mathrm{top} = \max\{r, \dim X\}$. Moreover, if $X = \mathbb{P}^3$ and $r = 2$, we will simply omitt them in the notation, that is, $\mathcal{M}(c_1, c_2, c_3)$ denotes the moduli space of rank 2 torsion free sheaves on $\mathbb{P}^3$ with Chern classes c_1, c_2 and c_3, $\mathcal{R}(c_1, c_2, c_3)$ the moduli space of reflexive sheaves, and $\mathcal{B}(c_1, c_2)$ the moduli space of locally free sheaves.

1.4 Monads

From now on, we will restrict our attention to sheaves on projective spaces. In these ambient spaces, monads are a very important tool to study locally free sheaves, and this section will be devoted to them.

Definition 47. A **monad** over a projective variety X is a complex,

$$M := 0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0 \tag{1.2}$$

of coherent sheaves over X which is exact at $\mathcal{A}$ and at $\mathcal{C}$, that means $ba = 0$, a is injective and b is surjective. The coherent sheaf $\mathcal{F} := \ker(b)/\mathrm{Im}(a)$ will be called **cohomology** of M and also denoted by $\mathrm{H}^\bullet(M) = \mathcal{F}$.

Definition 48. Let M be a monad of locally free sheaves on $\mathbb{P}^n$. The set

$$\Sigma(M) = \{x \in \mathbb{P}^n \mid a(x) \text{ is not injective}\}.$$

is called the **degeneracy locus** of the monad.

The following result will help us to determine whether the cohomology sheaf of a moanad is locally free, reflexive, or even torsion-free.

Lemma 49. Let

$$M := 0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0$$

be a monad of locally free sheaves on $\mathbb{P}^n$. Then

$$\Sigma(M) := \{x \in \mathbb{P}^n \mid \text{the stalk } \mathcal{F}_x \text{ is not a free } \mathcal{O}_{\mathbb{P}^n,x}\text{-module}\}.$$

Proof. See [20, Lemma 4]. □

The notion of monad is important in the study of vector bundles on $\mathbb{P}^n$ due to the following result of Horrocks.

Theorem 50. Let F be a rank r locally free sheaf on $\mathbb{P}^n$, then there is monad

$$M := 0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0$$

sucht that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are sums of line bundles, with $\operatorname{rk}\mathcal{B} - \operatorname{rk}\mathcal{A} - \operatorname{rk}\mathcal{C} = \operatorname{rk} F$.

Proof. See [29] . □

For an example of how the study of monads can help to describe the geometry of moduli spaces of locally free sheaves, the reader can check [55, Chapter 2].

Once we have defined the notion of monad, we would like to have the notion of morphisms between them.

Definition 51. Given two monads

$$M' := 0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0$$

$$M' := 0 \longrightarrow \mathcal{A}' \xrightarrow{a'} \mathcal{B}' \xrightarrow{b'} \mathcal{C}' \longrightarrow 0,$$

a **morphism of monads** is a triple of morphisms (f, g, h) such that the following diagram is commutative

$$\begin{tikzcd} M := 0 \arrow[r] & \mathcal{A} \arrow[r, "a"] \arrow[d, "f"'] & \mathcal{B} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{C} \arrow[r] \arrow[d, "h"'] & 0 . \\ M' := 0 \arrow[r] & \mathcal{A}' \arrow[r, "a'"] & \mathcal{B}' \arrow[r, "b'"] & \mathcal{C}' \arrow[r] & 0 \end{tikzcd}$$

In addition, if f, g and h are isomorphisms we say that the monads are **isomorphic**.

The following lemma gives a relation between isomorphism classes of monads and their cohomology when they are locally free; a proof can be found in [55, Lemma 4.1.3].

Lemma 52. Let E and E' be, respectively, locally free sheaves that are cohomology of the following monads:

$$M : \qquad A \xrightarrow{a} B \xrightarrow{b} C \tag{1.3}$$

$$M' : \qquad A' \xrightarrow{a'} B' \xrightarrow{b'} C' \tag{1.4}$$

If one has that $\mathrm{Hom}(B, A') = \mathrm{Hom}(C, B') = \mathrm{H}^1(X, C^\vee \otimes A') = \mathrm{H}^1(X, B^\vee \otimes A') = \mathrm{H}^1(X, C^\vee \otimes B') = \mathrm{H}^2(X, C^\vee \otimes A') = 0$ then there exists a bijection between the set of all morphisms from E to E' and the set of all morphisms of monads from (1.3) to (1.4).

The following important corollary will be used several times in what follows.

Corollary 53. Consider the monad

$$M : \qquad A \xrightarrow{a} B \xrightarrow{b} C$$

and its dual monad:

$$M^\vee : \qquad C^\vee \xrightarrow{b^\vee} B^\vee \xrightarrow{a^\vee} A^\vee .$$

If these monads satisfy the hypothesis of Lemma 52, and there exists an isomorphism $f : E \to E^\vee$ between its cohomology bundles such that $f^\vee = -f$, then there are isomorphisms $h : C \to A^\vee$, and $q : B \to B^\vee$, such that $q^\vee = -q$, and $h \circ b = a^\vee \circ q$.

Proof. See [55, Lemma 4.1.3, Corollary 2]. □

Given a locally free sheaf $\mathcal{F}$ on $\mathbb{P}^n$, sometimes it will be important to compute the dimension of the group $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$. Due to this, the following result is useful,

Lemma 54. Consider the monad

$$M : \qquad \mathcal{A} \xrightarrow{a_0} \mathcal{B} \xrightarrow{b_0} \mathcal{C}$$

with cohomology $\mathcal{E}$ being locally free. For the mapping

$$d_0 : \mathrm{Hom}(\mathcal{A}, \mathcal{B}) \oplus \mathrm{Hom}(\mathcal{B}, \mathcal{C}) \to \mathrm{Hom}(\mathcal{A}, \mathcal{C})$$

given by $d_0(a, b) = a_0 b + ab_0$.

If $\mathrm{Hom}(B, A') = \mathrm{Hom}(C, B') = \mathrm{H}^1(X, C^\vee \otimes A') = \mathrm{H}^1(X, B^\vee \otimes A') = \mathrm{H}^1(X, C^\vee \otimes B') = \mathrm{H}^2(X, C^\vee \otimes A') = 0$ then

$$\mathrm{coker} d_0 = \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$$

Proof. See [55, Lemma 4.1.7] □

1.5 Examples of Irreducible Components

In this section we will state some properties of two very important families of vector bundles: the instanton bundles and the generalized null correlation bundles.

Instanton bundles were extensively studied by the mathematical community in the past 60 years, and several interesting properties of them are known. They provide an example of irreducible and smooth component with the expected dimension for the moduli spaces $\mathcal{B}(0, c_2)$ for any c_2.

We now present the main results concerning instanton sheaves that will be used below. We start by recalling the definition of instanton sheaves on $\mathbb{P}^3$, cf. [33, Introduction] for further information on these objects.

Definition 55. An **instanton sheaf** on $\mathbb{P}^3$ is a torsion free coherent sheaf E with $c_1(E) = 0$ satisfying the following cohomological conditions:

$$\mathrm{H}^0(E(-1)) = \mathrm{H}^1(E(-2)) = \mathrm{H}^2(E(-2)) = \mathrm{H}^3(E(-3)) = 0.$$

The integer $c := c_2(E)$ is called the **charge** of E. When E is locally free, we say that E is an **instanton bundle**.

With this cohomological characterization it is possible to prove the following result.

Proposition 56. Any instanton sheaf on $\mathbb{P}^3$ with charge c and rank r can be obtained as the cohomology of monad of the form

$$0 \to c.\mathcal{O}_{\mathbb{P}^3}(-1) \to (r + 2c).\mathcal{O}_{\mathbb{P}^3} \to c.\mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

Theorem 57. Let $\mathcal{I}(c_2)$ be the family of rank 2 instanton bundles with second Chern class equals to c_2. Then $\mathcal{I}(c_2)$ fills out a smooth irreducible component of $\mathcal{B}(0, c_2)$ with dimension $8c_2 - 3$.

Proof. For the smoothness see [37], for the irreducibility see [61] and [62]. □

Recently, the study of moduli space of instanton bundles on $\mathbb{P}^3$ with rank $r \geqslant 3$ has attracted the interest of the mathematical community, as the reader can see for instance in[3] and [10]. Rank 4 instanton bundles will be particularly interesting in this work, and by this reason, we will gather here some important properties of them.

Lemma 58. (i) Every rank 4 instanton bundle E of charge 1 over $\mathbb{P}^3$ fits into an exact sequence:

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\mu} E \xrightarrow{\nu} N \to 0 \tag{1.5}$$

where N is a null correlation sheaf fitting into an exact triple:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s} \Omega^1_{\mathbb{P}^3}(1) \to N \to 0. \tag{1.6}$$

(ii) In addition, if N is locally free, then it is a null correlation bundle, and the triple (1.5) splits: $E \simeq N \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}$. Respectively, if N is not locally free, then it fits into an exact triple

$$0 \to N \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_l(1) \to 0, \tag{1.7}$$

where l is some projective line in $\mathbb{P}^3$.
(iii) There are exact triples induced by (1.5) and (1.6):

$$0 \to 3 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{S^2\mu} S^2E \to \operatorname{coker}(S^2\mu) \to 0, \qquad 0 \to 2 \cdot N \to \operatorname{coker}(S^2\mu) \to S^2N \to 0, \tag{1.8}$$

$$0 \to \wedge^2\Omega^1_{\mathbb{P}^3}(2) \xrightarrow{\zeta} \Omega^1_{\mathbb{P}^3}(1) \otimes N \xrightarrow{\eta} S^2N \to 0. \tag{1.9}$$

Proof. See [2, Lemma 4]. □

With this, it is possible to prove the following.

Corollary 59. In the conditions of Lemma 58, $h^0(S^2E) = 3,\ h^1(S^2E) = 5,\ h^2(S^2E) = 0.$

Proof. See [2, Corollary 5]. □

The Lemma 58 implies that in order to understand the rank 4 instanton bundles, we need to study the equations (1.7) and (3.3). Fixing a projective line $l \subset \mathbb{P}^3$, and let N_l denote the non locally free null correlation sheaf associated with l, as given in sequence (1.7). Note that

$$\dim \operatorname{Ext}^1(N_l, 2 \cdot \mathcal{O}_{\mathbb{P}^3}) = 2 \cdot h^2(N_l(-4)) = 2 \cdot h^1(\mathcal{O}_l(-3)) = 4,$$

so we must understand how many locally free extensions of N_l by $2 \cdot \mathcal{O}_{\mathbb{P}^3}$ do exist.

Lemma 60. For each line $l \subset \mathbb{P}^3$, the corresponding non locally free null correlation sheaf N_l admits an unique, up to isomorphism, locally free extension by $2 \cdot \mathcal{O}_{\mathbb{P}^3}$.

Proof. See [2, Lemma 6]. □

We conclude that the moduli space of rank 4 instanton bundles of charge 1, denoted by $\mathcal{I}(4,1)$, is isomorphic to $\mathbb{P}^5$. Indeed, let $G \subset \mathbb{P}^5$ denote the Grassmanian of lines in $\mathbb{P}^3$; the points in the complement $U := \mathbb{P}^5 \backslash G$ correspond to the **split instantons**, of the form $N \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}$; the points $l \in G$ correspond to the unique extention of N_l by $2 \cdot \mathcal{O}_{\mathbb{P}^3}$. These moduli spaces consider only isomorphism classes of instanton bundles, but if we are interested in their sympletic structure it will be important to determine how many sympletic structure each $E \in \mathcal{I}(4,1)$ has. The next lemma will answer this question.

Lemma 61. Every rank 4 instanton bundle E of charge 1 admits an unique symplectic structure, up to isomorphism.

Proof. See [2, Lemma 8] □

It is also possible to compute the structure of the automorphism group of any rank 4 instanton bundle, as we can see in the next lemma.

Lemma 62. If E is a rank 4 instanton bundle of charge 1, then $h^0(\mathrm{End}(E)) = 5$ and $\mathrm{Aut}(E) \simeq \mathbb{K}^* \times \mathrm{GL}(2)$.

Proof. See [2, Lemma 7] □

In [15], Ein studied families of locally free sheaves arising as cohomology of monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0$$

where $b \geqslant a \geqslant 0$ and $c > a + b$. Ein called such locally free sheaves of **Generalized null correlation bundles**. The following result will be used in this work.

Theorem 63. Let a, b and c be integers such that $b \geqslant a \geqslant 0$ and $c > a + b$. The family of all generalized null correlation bundles, arising as cohomology of the monad

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0$$

denoted by $\mathcal{E}(a,b,c)$, fullfills an irreducible smooth component of $\mathcal{B}(0,c^2-a^2-b^2)$, of dimension:

$$\binom{c+a+3}{3}+\binom{c+b+3}{3}+\binom{c-a+3}{3}+\binom{c-a+3}{3}+\binom{c-b+3}{3}$$
$$-\binom{a+b+3}{3}-\binom{b-a+3}{3}-\binom{2a+3}{3}-\binom{2b+3}{3}-3-\epsilon(a,b)$$

where $\epsilon(a,b)=4$ if $a=b=0$, or $\epsilon(a,b)=1$ if $a=0<b$ or $a=b>0$, or $\epsilon(a,b)=0$ if $0<a\leqslant b$.

Proof. See [15, Theorem 3.1] □

We call such irreducible components the **Ein components**. For a deeper study on the number of Ein components, or about their geometry the reader can see [42] and [43].

Another important tool to produce irreducible components in the moduli space of vector bundles is the Hartshorne-Serre correspondence, which we will now state.

Theorem 64 (Hartshorne-Serre Correspondence)**.** For any fixed line bundle L on $\mathbb{P}^3$, there is a bijective correspondence between (i) and (ii):

i) the set of triples $<\mathcal{E},s,\varphi>$ with the equivalence relation $\sim$, where $\mathcal{E}$ is a locally free sheaf of rank 2 on $\mathbb{P}^3$; $s\in \mathrm{H}^0(\mathcal{E})$ is a global section whose scheme of zeros is Y, has codimension 2; and $\varphi:\wedge^2\mathcal{E}\to L$ is an isomorphism. The equivalence relation $\sim$ is defined as follows: two triples $<\mathcal{E},s,\varphi>\sim<\mathcal{E}',s',\varphi'>$ if there is an isomorphism $\psi:\mathcal{E}\to\mathcal{E}'$, and a non-zero constant λ such that $s'=\lambda\psi(s)$, and $\varphi'=\lambda^2\varphi\circ(\wedge^2\varphi)^{-1}$.

ii) The set of pairs $<Y,\eta>$ where Y is a locally complete intersection curve in $\mathbb{P}^3$, and $\eta: L\otimes\omega_{\mathbb{P}^3}\otimes\mathcal{O}_Y\to\omega_Y$ is an isomorphism.

Proof. See [23, Theorem 1.1] □

In the context of the Theorem 64, we say that the locally free sheaf $\mathcal{E}$ comes from the extension of the curve Y. This is motivated by the fact that there is an exact sequence of the form:

$$0\to\mathcal{O}_{\mathbb{P}^3}(-c_1(\mathcal{E}))\to\mathcal{E}(-c_1(\mathcal{E}))\to I_Y\to 0$$

that relates $\mathcal{E}$ and I_Y where the later is the ideal sheaf of the curve Y.

Proposition 65. The set of rank 2 normalized stable locally free sheaves on $\mathbb{P}^3$ that came from the extension of r disjoint conics, lies inside an irreducible component of the moduli space $\mathcal{B}(-1,2(r-1))$ with dimension $8(2(r-1))-5$.

Proof. This is follows from [23, Proposition 4.1]. □

1.6 The spectrum of torsion free sheaves

In the previous sections we presented the problem of the study of the moduli spaces of torsion free sheaves on $\mathbb{P}^3$, and in the section 1.5 we presented some examples of irreducible componentes of these moduli spaces. In the section we will present the notion of spectrum of sheaves, which are a tool that help to compute the exact number of irreducible components of the moduli spaces. The notioin of spectrum for torsion free sheaves is due to Okonek and Spindler in [56].

Theorem 66. Let F be a rank 2 torsion free sheaf on $\mathbb{P}^3$, with generic splitting type (a_1, a_2), with $a_i \in \mathbb{Z}$, and $a_1 \leqslant a_2$. Let $s_F = h^0(\mathcal{E}xt^2(F, \mathcal{O}_{\mathbb{P}^3}))$ then there exists a list of m integers $(k_1, k_2, ..., k_m)$, with $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m$ such that

a) $h^1(F(l)) = s_F + \displaystyle\sum_{i=1}^{m} h^0(\mathcal{O}_{\mathbb{P}^1}(k_i + l + 1)$ if $l \leqslant a_2 - 1$

b $h^2(F(l)) = \displaystyle\sum_{i=1}^{m} h^1(\mathcal{O}_{\mathbb{P}^1}(k_i + l + 1)$ if $l \geqslant a_1 - 3$

Proof. See [56, Theorem 2.3]. □

Definition 67. Let F be a rank 2 torsion free sheaf on $\mathbb{P}^3$, the list of integers $(k_1, k_2, ..., k_m)$ given by the previous theorem is called **spectrum** of F.

For any torsion free sheaf E on $\mathbb{P}^3$, there exists a exact sequence of the form

$$0 \to R \to F \to E \to 0 \tag{1.10}$$

Where F is a locally free sheaf, and R a reflexive sheaf. Applying the functor $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^3})$ in the sequence (1.10), one has that $\mathcal{E}xt^2(E, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^1(R, \mathcal{O}_{\mathbb{P}^3})$. Therefore the support of the sheaf $\mathcal{E}xt^2(E, \mathcal{O}_{\mathbb{P}^3})$ is zero dimensional. Then it is possible to find a hyperplane $H \subset \mathbb{P}^3$ such that $H \cap \text{Supp } \mathcal{E}xt^2(E, \mathcal{O}_{\mathbb{P}^3}) = \varnothing$, and to define $s_{F_H} = h^0(\mathcal{E}xt^1(F, \mathcal{O}_{\mathbb{P}^3}) \otimes \mathcal{O}_H)$.

With this notation it is possible to obtain the following result, that give us numerical properties for the spectrum of a sheaf.

Proposition 68. Let F be a rank 2 torsion free sheaf on $\mathbb{P}^3$, with splitting type (a_1, a_2) and spectrum $(k_1, \ldots, k_m)$.

a) Let $k > a_2 + 1$, if there is at least $s_{F_H} + 1$ elements k_i in the spectrum, such that $k_i \geqslant k$, then each k', such that $a_2 + 1 \leqslant k' \leqslant k$ appears in the spectrum.

b) If $k \leqslant a_1 - 1$ is in the spectrum, so every integer $k^{'}$, such that $k \leqslant k^{'} \leqslant -1$. Is in the spectrum.

Proof. See [56, Prop 2.4]. □

For stable locally free sheaves, it is possible to obtain a better description for the spectrum.

Proposition 69. Let F be a rank 2 stable locally free sheaf on $\mathbb{P}^3$, with spectrum $(k_1, \ldots, k_m)$. Then the following claims are true:

a) $m = c_2(F)$;

b) 0 necessarily occurs in the spectrum;

c) If k_i is in the spectrum, then so is $-k_i$;

d) If $c_1 = 0$, then 0 or -1 appears in the spectrum at least twice.

Proof. See [25, Theorem 7.5] □

This proposition gives a sistematic way of studying all possible spectrum of stable locally free sheaves on $\mathbb{P}^3$ with fixed second Chern class, then in order to prove that for a given second Chern class, we have found all possible irreducible components, it is enough to check that these irreducible components contain the sheaves with all possible spectrum. The next couple of propositions will help us to understand better the spectrum of stable torsion free sheaves.

Proposition 70. Let E be a rank 2 torsion free sheaf on $\mathbb{P}^3$, with splitting type (a_1, a_2) and spectrum $(k_1, \ldots, k_m)$. If $a_2 - a_1 \leqslant 2$ then

$$\sum_{i=1}^{m} k_i = m(a_2 - 1) - \chi(F(-a_2 - 1)) - s_E$$

Proof. See [56, Proposition 2.6] □

Proposition 71. Let E be a torsion free sheaf on $\mathbb{P}^3$ with spectrum $K_F = (k_1, \ldots, k_m)$ $H \subseteq \mathbb{P}^3$ a general hyperplane, with splitting type (a_1, a_2) such that $a_1 - a_2 \leqslant 1$. Then we have:

$$m = \chi(F_H(-a_2 - 1)) \tag{1.11}$$

Proof. See [56, Proposition 2.7] □

We will use the above results to deduce some important properties of stable rank 2 torsion free sheaves in $\mathbb{P}^3$.

Proposition 72. Let E be a normalized semistable rank two torsion free sheaf on $\mathbb{P}^3$, with $c_3(E) = 0$, and $H \subseteq \mathbb{P}^3$ a generic hyperplane. Then one has

$$m = \chi(F_H(-1)) = c_2(E) \tag{1.12}$$

Proof. Since E is a normalized stable rank two torsion free sheaf on $\mathbb{P}^3$, we have that the splitting type of E is either $(-1, 0)$, if $c_1(E) = -1$ or $(0, 0)$, if $c_1(E) = 0$. By Proposition 71 is enough to prove that $c_2(E) = \chi(F_H(-1))$.

Fix a generic hyperplane $H \subset \mathbb{P}^3$. Consider the sequence of restriction

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_H \to 0 \tag{1.13}$$

Twisting this sequence by $E(-1)$, one has that

$$0 \to Tor_1(E, \mathcal{O}_H)(-1) \to E(-2) \to E(-1) \to E|_H(-1) \to 0 \tag{1.14}$$

Computing the Euler characteristic, we have that

$$\chi(E|_H(-1)) = \chi(E(-1)) - \chi(E(-2)) + \chi(Tor_1(E, \mathcal{O}_H)(-1)) = c_2(E) - h^0(Tor_1(E, \mathcal{O}_H)(-1)) \tag{1.15}$$

In the last inequality we use the fact that $Tor_1(E, \mathcal{O}_H)$ is a 0-dimensional sheaf. But using the semistability of E, and the sequence (1.14) one sees that $h^0(Tor_1(E, \mathcal{O}_H)(-1)) = 0$ what give us the result. $\square$

The next proposition characterizes the third Chern class of a torsion free sheaf in terms of its spectrum.

Proposition 73. Let E be a normalized, semistable torsion free sheaf on $\mathbb{P}^3$ with $s_E = h^0(\mathcal{E}xt^2(E, \mathcal{O}_{\mathbb{P}^3}))$. Then the following equalities are true.

a) If $c_1(E) = -1$, then $c_3(E) = -2\sum k_i - c_2(E) - 2s_E$.

b) If $c_1(E) = 0$, then $c_3(E) = -2\sum k_i - 2s_E$.

Proof. For the item $a)$, since $c_1((E) = -1$, recall that by Hirzenbruch-Riemann-Roch, the Euler Characteristic of E is $\chi(E(t)) = \dfrac{1}{6}(t+1)(t+2)(2t+3) - \dfrac{1}{2}(c_2(E)(2t+3) + c_3(E))$, using this, and the fact that $m = c_2(E)$ in Proposition 70 we have that $c_3(E) = -2\sum k_i - c_2(E) - 2s_E$ as we wanted.

The proof of item b) is analogous, just recall that the Euler Characteristic of E is $\chi(E(t)) = \dfrac{1}{3}(t+1)(t+2)(t+3) - (c_2(E)(t+2) + \dfrac{1}{2}c_3(E))$. □

It is interesting to note that Okonek and Spindler could prove the irreducibility of a large family of moduli spaces of torsion free sheaves on $\mathbb{P}^3$, [57], only using the notion of spectrum.

2 Moduli of locally free sheaves

As discussed in the Introduction, the problem of classification of vector bundles has, attracted the attention of the mathematical community. The problem that we will address in this chapter, was pointed, for instance at Harthsorne's list in [24, Problem 7], in his words: "Describe explicitly the moduli spaces $\mathcal{B}(c_1, c_2)$ of rank 2 stable bundles on $\mathbb{P}^3$, with Chern classes c_1 and c_2". It is surprisingly that, 40 years later this list come out, we only understand completely these moduli spaces for $c_2 \leqslant 4$.

In this chapter we will give examples of irreducible components of the moduli space of stable locally free sheaves on $\mathbb{P}^3$ different from those describe in the previous chapter. This new examples will help us to compute the exact number of irreducible components of $\mathcal{B}(0,5)$.

2.1 Modified instanton monads

An important object when studying locally free sheaves on projective spaces, is their first cohomology module.

Definition 74. Let E be a rank 2 locally free sheaf on $\mathbb{P}^3$. The set $\mathrm{H}^1_*(E) := \oplus_{t\in\mathbb{Z}}\mathrm{H}^1(E(t))$ has the structure of a graduated module, over the polynomial ring with 4 variables, and it is called **first cohomology module of** E.

The first cohomology module of vector bundles was the starting point of the study of the Ein Components in [15]; and it is possible to prove that the first cohomology module of any generalized null correlation bundle is generated by one element of degree $-a$ with $a < 0$. It is also possible to prove that the first cohomology module of any instanton bundle of charge k is generated by k elements of degree -1. A natural question that arises here is: what can be said of locally free sheaves whose first cohomolgy module is generated by one element of degree $-a$ and k elements of degree -1? We will dedicate the rest of this chapter to the answer.

For $a \geqslant 2$ and $k \geqslant 1$, consider the following monad:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{2.1}$$

which we call **modified instanton monads**. The family of isomorphism classes of bundles arising as cohomology of such monads will be denoted by $\mathcal{G}(a,k)$.

Proposition 75. A vector bundle E on $\mathbb{P}^3$ is the cohomology of a monad of the form (2.1) if and only if $\mathrm{H}^1_*(E)$ has one generator in degree $-a$ and k generators in degree -1, and its Chern classes are $c_1(E) = 0$, and $c_2(E) = a^2 + k$.

Proof. The "only if" part is straightforward. Indeed, if E is cohomology of a monad of the form (2.1), then, by the display of the monad, $\mathrm{H}^1_*(E)$ has one generator in degree $-a$ and k generators in degree -1. On the other hand, if $c_1(E) = 0$, it implies that E is a self dual vector bundle on $\mathbb{P}^3$, additionally, if $\mathrm{H}^1(E)$ has one generator in degree $-a$ and k generators in degree -1, then by [38, Theorem 2.3], E is cohomology of a monad of the type:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \oplus_{i=1}^{2k+4}\mathcal{O}_{\mathbb{P}^3}(k_i) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

Computing the Chern class give us $c_2(E) = a^2 + k - \sum_{i=1}^{6} k_i^2$, since $c_2(E) = a^2 + k$, we have $k_i = 0$ for all i. □

Note that, by now, $\mathcal{G}(a, k)$ could possibly be empty. The next proposition shows that this is not the case.

Proposition 76. For each $a \geqslant 2$ and $k \geqslant 1$, the family $\mathcal{G}(a, k)$ is non-empty and contains stable bundles, while every $E \in \mathcal{G}(a, k)$ is μ-semistable. In addition, every $E \in \mathcal{G}(a, 1)$ is stable.

Proof. First, we will prove that there exists a $E \in \mathcal{G}(a, k)$ such that E is stable. Let F be a rank 2 instanton bundle of charge k. Let $a \geqslant 2$ and take $\sigma \in \mathrm{H}^0(F(2a))$ and cosider the zero locus of σ, $(\sigma)_0 = X$ (such σ always exists if F is a 't Hooft instanton bundle, for instance). Let Y be a complete intersection of two surfaces of degree a and $X \cap Y = \varnothing$. According to [27, Lemma 4.8], there exists a bundle E and a section $\tau \in \mathrm{H}^0(E(a))$ such that $(\tau)_0 = Y \cup X$ which is given as cohomology of a monad of the form (2.1). In addition, since F is stable, X is not contained in any surface of degree a, hence neither is $Y \cup X$, and E is also stable.

It is straightforward to check that every $E \in \mathcal{G}(a, k)$ satisfies $h^0(E(-1)) = 0$, thus E is μ-semistable.

Now, we will prove that every $E \in \mathcal{G}(a, 1)$ is stable. Fix $k = 1$, and assume that there is $E \in \mathcal{G}(a, 1)$ satisfying $h^0(E) \neq 0$. Setting $K := \ker\beta$, it follows that $h^0(K) \neq 0$, hence the quotient $K' := K/\mathcal{O}_{\mathbb{P}^3}$ fits into the following exact sequence

$$0 \to K' \to 5 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta'} \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0.$$

hence $(K')^\vee$ admits a resolution of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{(\beta')^\vee} 5 \cdot \mathcal{O}_{\mathbb{P}^3} \to (K')^\vee \to 0.$$

By [7, Theorem 2.7] $(K')^\vee$ is μ-stable, hence K' is μ-stable. However, the monomorphism $\alpha : \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to K$ induces a monomorphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to K'$ (because $K' := K/\mathcal{O}_{\mathbb{P}^3}$); by the μ-stability of K', we should have

$$-1 < \mu(K') = -\frac{a+1}{3} \implies a < 2,$$

providing the desired contradiction. □

The modified instanton bundles are also related to usual instanton bundles of higher rank in a very important way. The precise relationship is outlined in the next couple of lemmas, and then summarized in Proposition 80 below.

Lemma 77. Given a vector bundle $E \in \mathcal{G}(a,k)$, there exists a rank 4 instanton bundle $\tilde{E}$ of charge k, and sections $\sigma \in H^0(\tilde{E}(a))$, $\tau \in H^0(\tilde{E}^\vee(a))$ such that the complex:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0 \tag{2.2}$$

is a monad whose cohomology is isomorphic to E.

Proof. Define $\tilde{\alpha} = \alpha \circ i$ and $\tilde{\beta} = \pi \circ \beta$ where $i : k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1)$ is the inclusion and $\pi : \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to k \cdot \mathcal{O}_{\mathbb{P}^3}(1)$ is the projection. It is clear that $\tilde{\alpha}$ is injective and $\tilde{\beta}$ is surjective. We then get the following monad, whose cohomology is a rank 4 instanton $\tilde{E}$ of charge k:

$$0 \to k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\beta}} k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{2.3}$$

Now we need to construct the morphisms σ and τ. It is straightforward to check that the chain of inclusions: $\operatorname{im} \tilde{\alpha} \subseteq \operatorname{im} \alpha \subseteq \ker \beta \subseteq \ker \tilde{\beta}$ holds. For this reason, we have:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathcal{O}_{\mathbb{P}^3}(a) \arrow[d] & \\
0 \arrow[r] & \ker \beta \arrow[r] \arrow[d, "f_1"] & (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \tilde{\beta} \arrow[r, "i"] & (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0, &
\end{tikzcd}$$

where f_1 is the inclusion. It follows that coker $f_1 \simeq \mathcal{O}_{\mathbb{P}^3}(a)$. In addition, we also obtain the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \operatorname{im}\tilde{\alpha} \arrow[r, "i"] \arrow[d, equal] & \ker\beta \arrow[r] \arrow[d, "f_1"] & \ker\beta/\operatorname{im}\tilde{\alpha} \arrow[r] \arrow[d, "\omega"] & 0 \\
0 \arrow[r] & \operatorname{im}\tilde{\alpha} \arrow[r, "i"] & \ker\tilde{\beta} \arrow[r] \arrow[d] & \tilde{E} \arrow[r] & 0 \\
 & & \mathcal{O}_{\mathbb{P}_3}(a) \arrow[d] & & \\
 & & 0, & &
\end{tikzcd}$$

where ω is the inclusion. Thus coker $\omega \simeq \mathcal{O}_{\mathbb{P}^3}(a)$, and we obtain an epimorphism $\tau : \tilde{E} \to \mathcal{O}_{\mathbb{P}^3}(a)$.

Now, by the isomorphism theorem we have $\dfrac{\ker\beta}{\operatorname{im}\alpha} \simeq \dfrac{\frac{\ker\beta}{\operatorname{im}\tilde{\alpha}}}{\frac{\operatorname{im}\alpha}{\operatorname{im}\tilde{\alpha}}}$, so there exists an epimorphism $f_2 : \ker\beta/\operatorname{im}\tilde{\alpha} \to E$ fitting into the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & k\cdot\mathcal{O}_{\mathbb{P}_3}(-1) \arrow[r, "\tilde{\alpha}"] \arrow[d] & \ker\beta \arrow[r] \arrow[d, equal] & \ker\beta/\operatorname{im}\tilde{\alpha} \arrow[r] \arrow[d, "f_2"] & 0 \\
0 \arrow[r] & k\cdot\mathcal{O}_{\mathbb{P}_3}(-1)\oplus\mathcal{O}_{\mathbb{P}_3}(-a) \arrow[r] \arrow[d] & \ker\beta \arrow[r] & E \arrow[r] \arrow[d] & 0 \\
 & \mathcal{O}_{\mathbb{P}_3}(-a) \arrow[d] & & 0 & \\
 & 0. & & &
\end{tikzcd}$$

It follows that $\ker f_2 \simeq \mathcal{O}_{\mathbb{P}_3}(-a)$, so there exists a monomorphism $\sigma' : \mathcal{O}_{\mathbb{P}^3}(-a) \to \ker\beta/\operatorname{im}\tilde{\alpha}$. Composing it with ω, we obtain a monomorphism $\sigma := \omega\circ\sigma' : \mathcal{O}_{\mathbb{P}^3}(-a) \to \tilde{E}$. An epimorphism τ is constructed in a similar way. We have therefore constructed the monad:

$$0 \to \mathcal{O}_{\mathbb{P}_3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}_3}(a) \to 0,$$

whose cohomology is precisely the bundle E (note that the leftmost row in the above diagramm gives E in the display of such monad). $\square$

Lemma 78. If a bundle E is the cohomology of a monad of the form (2.2), in which $\tilde{E}$ is a sympletic rank 4 instanton bundle, then E is also isomorphic to the cohomology of a monad of the form (2.1), i.e. $E \in \mathcal{G}(a, k)$.

Proof. Let $\tilde{E}$ be an rank 4 instanton bundle of charge k over $\mathbb{P}^3$, so that $\tilde{E}$ is cohomology of a monad of the type:

$$0 \to k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\beta}} k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

Take $\tau, \sigma \in \mathrm{H}^0(\tilde{E}(a))$ satisfying $\tau \circ \sigma = 0$. We thus have the following exacts sequences:

$$0 \to \ker \tilde{\beta} \to (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\beta}} k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

$$0 \to k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} \ker \tilde{\beta} \to \tilde{E} \to 0,$$

$$0 \to \ker \tau \to \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \ker \tau \to E \to 0,$$

where $E \simeq \ker \tau / \operatorname{im} \sigma$.

First, define a morphism $f_2 : \ker \tau \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to \ker \tilde{\beta}$ as follows: given x and y local sections of $\ker \tau$ and $k \cdot \mathcal{O}_{\mathbb{P}^3}(-1)$, respectively, we set $f_2(x, y) := x + \tilde{\alpha}(y)$ (note that f_2 is well defined since $\tilde{\alpha}$ is injective), where x in the right hand side of the equality is regarded as a local section of $\tilde{E} \simeq \ker \tilde{\beta} / \operatorname{im} \tilde{\alpha}$. We thus obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d, equal] & \ker \tau \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d, "f_2"] & \ker \tau \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "\tilde{\alpha}"] & \ker \tilde{\beta} \arrow[r] & \tilde{E} \arrow[r] & 0,
\end{tikzcd}$$

from which we obtain the exact sequence:

$$0 \to \ker \tau \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{f_2} \ker \tilde{\beta} \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0.$$

We can then compose f_2 with the inclusion $\ker \tilde{\beta} \subseteq (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3}$, obtaining a monomorphism $\tilde{f}_2$ fitting into the diagram below:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \ker \tau \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, equal] \arrow[d, "f_2"] & \ker \tau \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[d, "\tilde{f}_2"] & & \\
0 \arrow[r] & \ker \tilde{\beta} \arrow[r, "i"] \arrow[d] & (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \arrow[r, "\tilde{\beta}"] \arrow[d] & k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(a) \arrow[r] \arrow[d] & \operatorname{coker} \tilde{f}_2 \arrow[r] \arrow[d] & k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with the third line obtained via the Snake Lemma; it follows that coker $\tilde{f}_2 \simeq \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1)$.

Let $\beta : (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1)$ denote the natural quotient morphism. Making

$$\alpha := \tilde{f}_2 \circ (\sigma, \mathbf{1}_{k \cdot \mathcal{O}_{\mathbb{P}^3}(-1)}) : \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3},$$

we get the monad:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

whose cohomology is isomorphic to E. □

Next, we argue that the instanton bundle $\tilde{E}$ obtained in Proposition 77 is symplectic.

Lemma 79. If $\tilde{E}$ is a rank 4 instanton bundle of charge k that fits in a monad of the form (2.2), such that the cohomology is a vector bundle, then $\tilde{E}$ admits a symplectic structure, and τ is determined by σ.

Proof. Since E is a rank 2 vector bundle with $c_1(E) = 0$, there is a (unique up to scale) symplectic isomorphism $\varphi : E \xrightarrow{\simeq} E^\vee$. By Corollary 53, there is an isomorphism of monads:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-a) \arrow[r, "\sigma"] \arrow[d, "\simeq"', "g"] & \tilde{E} \arrow[r, "\tau"] \arrow[d, "\simeq"', "\varphi"] & \mathcal{O}_{\mathbb{P}^3}(a) \arrow[r] \arrow[d, "\simeq"', "h"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-a) \arrow[r, "\tau^\vee"] & \tilde{E}^\vee \arrow[r, "\sigma^\vee"] & \mathcal{O}_{\mathbb{P}^3}(a) \arrow[r] & 0
\end{tikzcd}$$

such that $\varphi^\vee = -\varphi$, so $(\tilde{E}, \varphi)$ is a symplectic instanton bundle, and $\tau = \sigma^\vee \circ \varphi$. □

Gathering the Lemmas 77, 78 and 79, we obtain the following statement.

Proposition 80. A rank 2 bundle E is the cohomology of a monad of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

if and only if it is also the cohomology of a monad of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\sigma^\vee \circ \varphi} \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

where $(\tilde{E}, \varphi)$ is a rank 4 symplectic instanton bundle of charge k.

As a first application of Proposition 80 we provide an alternative, more manageable description of the set $\mathcal{G}(a,k)$.

In order to fix the notation, note that every automorphism $f \in \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1))$ can be represented by a $(k+1) \times (k+1)$ matrix :

$$f = \begin{pmatrix} f_{1,1} & 0 & \cdots & 0 \\ f_{2,1} & f_{2,2} & \cdots & f_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k+1,1} & f_{k+1,2} & \cdots & f_{k+1,k+1} \end{pmatrix}$$

where each $f_{j,1} \in H^0(\mathcal{O}_{\mathbb{P}^3}(a-1))$ with $j = 2, \ldots, k+1$, and $f_{1,1}$ and $f_{l,m}$ are constants for $l, m = 2, 3, \ldots, (k+1)$ such that:

$$f_{1,1} \cdot \det \begin{pmatrix} f_{2,2} & \cdots & f_{2,k+1} \\ \vdots & \ddots & \vdots \\ f_{k+1,2} & \cdots & f_{k+1,k+1} \end{pmatrix} \neq 0.$$

We will denote:

$$\tilde{f} = \begin{pmatrix} f_{2,2} & \cdots & f_{2,k+1} \\ \vdots & \ddots & \vdots \\ f_{k+1,2} & \cdots & f_{k+1,k+1} \end{pmatrix};$$

clearly, $\tilde{f} \in \mathrm{Aut}(k \cdot \mathcal{O}_{\mathbb{P}^3}(-1))$.

Similarly, every $h \in \mathrm{Aut}(k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a))$ can be represented by a $(k+1) \times (k+1)$ matrix:

$$M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,k} & 0 \\ M_{2,1} & \cdots & M_{2,k} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{k+1,1} & M_{k+1,2} & \cdots & M_{k+1,k+1} \end{pmatrix}$$

where each $M_{k+1,j} \in H^0(\mathcal{O}_{\mathbb{P}^3}(a-1))$ for $j = 1, \ldots, k$, and $M_{k+1,k+1}$ and $h_{l,m}$ are constants for $l, m = 1, 2, \ldots, k$, such that:

$$M_{k+1,k+1} \cdot \det \begin{pmatrix} M_{1,1} & \cdots & M_{1,k} \\ \vdots & \ddots & \vdots \\ M_{k,1} & \cdots & M_{k,k} \end{pmatrix} \neq 0.$$

We will denote:

$$\tilde{M} = \begin{pmatrix} M_{1,1} & \cdots & M_{1,k} \\ \vdots & \ddots & \vdots \\ M_{k,1} & \cdots & M_{k,k} \end{pmatrix}.$$

Clearly, $\tilde{M} \in \mathrm{Aut}(k \cdot \mathcal{O}_{\mathbb{P}^3}(1))$.

Now let $\mathcal{P}(a,k)$ be the set of pairs $((\tilde{E},\varphi),\sigma)$ consisting of a rank 4 symplectic instanton bundle $(\tilde{E},\varphi)$ of charge k, and a nowhere vanishing section $\sigma \in \mathrm{H}^0(\tilde{E}(a))$, equipped with the following equivalence relation: $((\tilde{E},\varphi),\sigma) \sim ((\tilde{E}',\varphi'),\sigma')$ if and only if there are an isomorphism of symplectic bundles $g : (\tilde{E},\varphi) \xrightarrow{\sim} (\tilde{E}',\varphi')$, and a constant $\lambda \in \mathbb{K}^*$ such that $g \circ \sigma = \lambda\sigma$. We will denote each equivalence class in $\mathcal{P}(a,k)$ by $[(\tilde{E},\varphi),\sigma]$.

Theorem 81. There exists a bijection between $\mathcal{G}(a,k)$ and $\mathcal{P}(a,k)$.

Proof. From each pair $((\tilde{E},\varphi),\sigma)$ we build the monad

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\sigma^\vee \circ \varphi} \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

whose cohomology, by Proposition 80, yields an element $[E] \in \mathcal{G}(a,k)$. Two equivalent pairs $((\tilde{E},\varphi),\sigma)$ and $((\tilde{E}',\varphi'),\sigma')$ yield isomorphic monads

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-a) \arrow[r, "\sigma"] \arrow[d, "\lambda"] & \tilde{E} \arrow[r, "\sigma^\vee \circ \varphi"] \arrow[d, "g"] & \mathcal{O}_{\mathbb{P}^3}(a) \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-a) \arrow[r, "\sigma'"] & \tilde{E}' \arrow[r, "\sigma'^\vee \circ \varphi'"] & \mathcal{O}_{\mathbb{P}^3}(a) \arrow[r] & 0,
\end{tikzcd}$$

thus $[E] = [E']$.

Conversely, any $[E] \in \mathcal{G}(a,k)$ is the cohomology of a monad of the form (2.1), from which we can obtain, via Proposition 80, a pair $((\tilde{E},\varphi),\sigma)$. Any two monads whose cohomologies are isomorphic to E are also isomorphic, by Lemma 52; since E is rank 2 vector bundle with zero first Chern class, then Corollary 53 implies the existence of a skew symmetric isomorphism of monads:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "\alpha"] \arrow[d, "f"] & (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \arrow[r, "\beta"] \arrow[d, "g"] & \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d, "M"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "\alpha'"] & (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \arrow[r, "\beta'"] & \mathcal{O}_{\mathbb{P}^3}(a) \oplus k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] & 0.
\end{tikzcd}$$

It then follows that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{f}"] & (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \arrow[r, "\tilde{\beta}"] \arrow[d, "g"] & k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d, "\tilde{M}"] & 0 \\
0 \arrow[r] & k \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "\tilde{\alpha}'"] & (4+2k) \cdot \mathcal{O}_{\mathbb{P}^3} \arrow[r, "\tilde{\beta}'"] & k \cdot \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] & 0
\end{tikzcd}$$

provides an isomorphism of monads, since $\tilde{f}, g, \tilde{M}$ are isomorphisms, which in turn induces an isomorphism $g : \tilde{E} \xrightarrow{\sim} \tilde{E}'$.

In addition, we also have the following isomorphism of monads

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-a) \arrow[r, "\sigma"] \arrow[d, "f_{1,1}"] & \tilde{E} \arrow[r, "\tau"] \arrow[d, "g"] & \mathcal{O}_{\mathbb{P}^3}(a) \arrow[r] \arrow[d, "M_{k+1,k+1}"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-a) \arrow[r, "\sigma'"] & \tilde{E}' \arrow[r, "\tau'"] & \mathcal{O}_{\mathbb{P}^3}(a) \arrow[r] & 0, \end{tikzcd} \tag{2.4}$$

which implies that $g\sigma = f_{1,1} \cdot \sigma'$.

Corollary 53 tells us that $\tilde{E}$ and $\tilde{E}'$ admit symplectic structures φ and φ', respectively, and it only remains for us to show that $(\tilde{E}, \varphi)$ and $(\tilde{E}', \varphi')$ are isomorphic as symplectic bundles. By Lemma 79, one can take $\tau = \sigma^\vee \circ \varphi$ and $\tau' = \sigma'^\vee \circ \varphi'$ in equation (2.4), so that the commutation of the right square in that diagram yields $\sigma'^\vee \circ \varphi' = h_{k+1,k+1} \cdot \sigma^\vee \circ \varphi$. Since $\sigma'^\vee = f_{1,1}^{-1} \cdot \sigma^\vee \circ g^\vee$, we conclude that $f_{1,1} = h_{k+1,k+1}$ and $g^\vee \circ \varphi' \circ g = \varphi$, as desired. □

For our second application of Proposition 80, we focus our attention on the case $k = 1$ to obtain the following important formula for the case $k = 1$.

Lemma 82. For every $E \in \mathcal{G}(a, 1)$ with $a \geqslant 2$, the following holds

$$\mathrm{h}^1(\mathrm{End}(E)) = 4 \cdot \binom{a+3}{3} - a - 1 + \varepsilon(a),$$

where $\varepsilon(a) = 1$ when $a = 3$, and $\varepsilon(a) = 0$ when $a \neq 3$.

Proof. See [2, Lemma 16] □

It is interesting to observe that the right hand side of the formula in Lemma 82 yields the expected value when $a = 2$ and $a = 3$, respectively 37 and 77; when $a \geqslant 4$, one can check that $4 \cdot \binom{a+3}{3} - a - 1 > 8(a^2+1) - 3$.

2.2 The structure of $\mathcal{P}(a, 1)$

Motivated by Lemma 82, we now aim at showing that the set $\mathcal{P}(a, 1)$ has the structure of an irreducible, nonsingular, quasi-projective variety whose dimension matches the formula in the statement of the lemma. We will not make any distinction between a vector bundle E and its isomorphism class $[E]$ and will denote both of them by the letter E without the brackets.

Recall that a null correlation bundle is, by definition, the cokernel of a nonzero morphism $\eta \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-1), \Omega_{\mathbb{P}^3}(1))$ up to a scalar factor, so that the moduli space of null correlation sheaves can be identified with $\mathbb{P}(\mathrm{H}^0(\Omega_{\mathbb{P}^3}(2))) \simeq \mathbb{P}^5$. Denoting by N_η the null correlation sheaf defined by $\eta \in \mathbb{P}(\mathrm{H}^0(\Omega_{\mathbb{P}^3}(2)))$, we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(a-1)) \xrightarrow{\eta} \Omega_{\mathbb{P}^3}(a+1) \longrightarrow N_\eta(a) \longrightarrow 0.$$

Therefore, by the long exact sequence of cohomology, there exists a natural isomorphism of $\mathrm{H}^0(N_\eta(a))$ with the quotient vector space $\mathrm{H}^0(\Omega_{\mathbb{P}^3}(a+1))/\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a-1))$.

Setting $V := \mathbb{P}(\mathrm{H}^0(\Omega_{\mathbb{P}^3}(2)))$, consider the morphism

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a-1)) \otimes \mathcal{O}_V(-1) \xrightarrow{\tilde{\eta}} \mathrm{H}^0(\Omega_{\mathbb{P}^3}(a+1)) \otimes \mathcal{O}_V$$

given by multiplication by the coordinates. This is clearly injective, and its cokernel is a vector bundle over V, denoted by $\mathbf{N}_a$, whose fibre over $\eta \in V$ is $\mathrm{H}^0(\operatorname{coker} \eta(a)) \simeq \mathrm{H}^0(N_\eta(a))$.

From Lemmas 58 and 60, we know that each rank 4 instanton bundle $\tilde{E}$ of charge 1 corresponds to a unique null correlation sheaf $N := \tilde{E}/2 \cdot \mathcal{O}_{\mathbb{P}^3}$.

Note that the exact sequence (2.2) yields an exact sequence in cohomology for every $a \geqslant 1$:

$$0 \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a))^{\oplus 2} \to \mathrm{H}^0(\tilde{E}(a)) \to \mathrm{H}^0(N(a)) \to 0.$$

It follows that

$$\mathrm{H}^0(\tilde{E}(a)) \simeq \mathrm{H}^0(N(a)) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a))^{\oplus 2}, \tag{2.5}$$

so every section $\sigma \in \mathrm{H}^0(\tilde{E}(a))$ can be represented as a triple $(\sigma_N, \sigma_1, \sigma_2)$ with $\sigma_N \in \mathrm{H}^0(N(a))$ and $\sigma_1, \sigma_2 \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a))$. In this representation, the action of $\mathrm{Aut}(E)$ on $\mathrm{H}^0(\tilde{E}(a))$ is given by

$$(\lambda, M) \cdot (\sigma_N, \sigma_1, \sigma_2) = (\lambda \cdot \sigma_N, \sigma_1', \sigma_2'), \quad \text{where} \quad \begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = M \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}. \tag{2.6}$$

Since $\tilde{E}$ admits a unique symplectic structure, the splitting in cohomology given in equation (2.5) implies that any pair $((\tilde{E}, \varphi), \sigma)$, consisting of a sympectic rank 4 instanton bundle of charge 1 and a section $\sigma \in \mathrm{H}^0(\tilde{E}(a))$, can be regarded as a point of the product $\mathbf{N}_a \times \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a))^{\oplus 2}$, namely $((N, \sigma_N), (\sigma_1, \sigma_2))$ in the notation of equation (2.6).

Moreover, equation (2.6) also implies that two equivalent pairs $((\tilde{E}, \varphi), \sigma)$ and $((\tilde{E}', \varphi'), \sigma')$ will correspond to points $((N, \sigma_N), (\sigma_1, \sigma_2))$ and $((N, \lambda\sigma_N), (\sigma_1', \sigma_2'))$, respectively, where

$$\lambda \begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = M \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix};$$

here, (λ, M) is the pair representing the symplectic isomorphism $(\tilde{E}, \varphi) \xrightarrow{\sim} (\tilde{E}', \varphi')$ under the isomorphism of Lemma 62. In other words, an equivalence class $[(\tilde{E}, \varphi), \sigma] \in \mathcal{P}(a, 1)$ defines a unique point in the Grassmannian $G(2, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a)))$ of 2-dimensional subspaces of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(a))$.

Proposition 83. $\mathcal{P}(a, 1)$ is an irreducible, rational, nonsingular quasi-projective variety of dimension

$$5 + h^0(N(a)) + 2 \cdot (h^0(\mathcal{O}_{\mathbb{P}^3}(a)) - 2) = 4 \cdot \binom{a+3}{3} - a - 1.$$

Proof. We start by defining the following map, using the notation of the previous paragraph:

$$\pi : \mathcal{P}(a,1) \to \mathbb{P}^5 \times G(2, h^0(\mathcal{O}_{\mathbb{P}^3}(a)))$$
$$[(\tilde{E}, \varphi), \sigma] \mapsto (\tilde{E}/2 \cdot \mathcal{O}_{\mathbb{P}^3}, \langle \sigma_1, \sigma_2 \rangle).$$

This is clearly well defined, and we check that it is surjective. Given a null correlation sheaf $N \in \mathbb{P}^5$, let $\tilde{E}$ be the unique locally free extension of N by $2 \cdot \mathcal{O}_{\mathbb{P}^3}$, and let φ be its unique symplectic structure.

Next, take $\langle \sigma_1, \sigma_2 \rangle \in G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(a)))$, and note that the set $\{\sigma_1 = \sigma_2 = 0\}$ is a complete intersection curve C (of degree a^2) in $\mathbb{P}^3$. One can find a section $\sigma_N \in H^0(N(a))$ whose zero locus, being a curve of degree $a^2 + 1$, does not intersect C. The triple $(\sigma_N, \sigma_1, \sigma_2)$ thus obtained defines a nowhere vanishing section $\sigma \in H^0(\tilde{E}(a))$.

Clearly, the set $\pi^{-1}(N, \langle \sigma_1, \sigma_2 \rangle)$ consists of all those sections $\sigma_N \in H^0(N(a))$ which do not vanish along the curve $C := \{\sigma_1 = \sigma_2 = 0\}$, so it is an open subset of $H^0(N(a))$. It follows that $\mathcal{P}(a,1)$ can be regarded as an open subset of the product $\mathbf{N}_a \times G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(a)))$, showing that $\mathcal{P}(a,1)$ is an irreducible, nonsingular quasi-projective variety of the given dimension.

Finally, note that $\mathbf{N}_a$ is rational, since it is the total space of a vector bundle over $\mathbb{P}^5$. Hence the product $\mathbf{N}_a \times G(2, H^0(\mathcal{O}_{\mathbb{P}^3}(a)))$ is rational, and so is $\mathcal{P}(a,1)$. □

Noting that the dimension of $\mathcal{P}(a,1)$ matches $h^1(\mathrm{End}(E))$ for $a = 2$ and $a \geqslant 4$, as calculated in Lemma 82, we have therefore completed the proof of the first main result of this work.

Theorem 84. For $a = 2$ and $a \geqslant 4$, the rank 2 bundles given as cohomology of monads of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

fill out an open subset of an irreducible component of $\mathcal{B}(0, a^2 + 1)$ of dimension

$$4 \cdot \binom{a+3}{3} - a - 1.$$

Proof. See [2, Theorem 18] □

We will present here the proof for particular case.

Proposition 85. The moduli space $\mathcal{B}(0,5)$ contains an irreducible component of dimension 37, whose generic sheaf is cohomology of the monad

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} 6.\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{2.7}$$

Proof. Consider for each point $\mathbf{t} \in \mathcal{P}(2,1)$ the locally free sheaf $E(\mathbf{t}) \in \mathcal{B}(0,5)$. By construction, this give us a morphism from $\mathcal{P}(2,1) \to \mathcal{B}(0,5)$. For $\mathbf{t}_0$, let $E(\mathbf{t}_0)$ be the locally free sheaf which is the cohomology of the monad (2.1) given by the maps:

$$\alpha = \begin{pmatrix} y^2 & w^2 \\ -x^2 & -z^2 \\ -zy & 0 \\ wx & 0 \\ z^2 & y^2 \\ -w^2 & -x^2 \end{pmatrix}, \tag{2.8}$$

$$\beta = \begin{pmatrix} x^2 & y^2 & z^2 & w^2 & zy & wx \\ z^2 & w^2 & y^2 & x^2 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

One easily checks that $\beta \circ \alpha = 0$; moreover, note that the following minors of both matrices

$$y^3 - w^2 z \quad , \quad xy^2 - z^3 \quad , \quad x^3 - z^2 w$$

$$yx^2 - w^3 \ \text{ and } \ x^2yz^2$$

do not vanish simultaneously. It follows that the cohomology sheaf is locally free.

Using the notation of the Lemma 54, let $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Fixing basis for $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1), 6.\mathcal{O}_{\mathbb{P}^3})$, for $\mathrm{Hom}(6.\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1))$ and for $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)))$ and applying d_0 in each element of the fixed basis of $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1), 6.\mathcal{O}_{\mathbb{P}^3}) \oplus \mathrm{Hom}(6.\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1))$, one finds the matrix representation for d_0 (which is a matrix of order 85×168) and using Macaulay2 (or any software for computations with matrices), we check that d_0 is surjective, hence $\mathrm{Ext}^2(E(\mathbf{t}), E(\mathbf{t})) = \mathrm{coker}\, d_0 = 0$.

By the stability of $E(\mathbf{t})$, one has $\dim \mathrm{Ext}^1(E(\mathbf{t}), E(\mathbf{t})) = \dim \mathcal{P}(2,1)$. Since $\mathcal{P}(2,1)$ is irreducible, there exists an open dense in $U \subseteq \mathcal{P}(2,1)$, such that for each $\mathbf{t} \in U$, one has $\dim \mathrm{Ext}^1(E(\mathbf{t}), E(\mathbf{t})) = \dim U$. This implies that, the closure of the image of U, hence the closure of the image $\mathcal{P}(2,1)$ into $\mathcal{B}(0,5)$ is an irreducible component of dimension 37. $\square$

2.3 Components of $\mathcal{B}(0,5)$

As we discussed in the Introduction, the smallest integer n such that all the irreducible components of $\mathcal{B}(0,n)$ is not known is $n = 5$. In this section we will use the Theorem 84 to study the number of irreducible components of $\mathcal{B}(0,5)$.

From the possible spectrums of the locally free sheaves in $\mathcal{B}(0,5)$, Hartshorne and Rao, proved that every bundle there is cohomology of one of the following monads, cf. [27, Table 5.3, pages 803–804].

$$0 \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to 12 \cdot \mathcal{O}_{\mathbb{P}^3} \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \quad \text{and} \tag{2.10}$$

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \to 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 3 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(2) \to 0; \tag{2.11}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0; \tag{2.12}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \quad \text{and} \tag{2.13}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 6 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{2.14}$$

Recall that for each stable rank 2 bundle E on $\mathbb{P}^3$ with vanishing first Chern class, the number $\alpha(E) := h^1(E(-2)) \bmod 2$ is called the Atiyah–Rees α-invariant of E, see [23, Definition in page 237]. Hartshorne showed [23, Corollary 2.4] that this number is invariant on the components of the moduli space of stable vector bundles on $\mathbb{P}^3$. One can easily check that the cohomologies of monads of the form (2.10) and (2.11) have α-invariant equal to 0, while the cohomologies of the other three types of monads have α-invariant equal to 1.

Our first step here will be to prove that the family of stable locally free sheaves which are cohomology of the monad 2.14 are contained in the closure of $\mathcal{G}(2,1)$. Precisely, consider the set:

$$\mathcal{H} = \{[E] \in \mathcal{B}(0,5) \mid E \ \text{ is cohomology of a monad of the form (2.14)}\}.$$

then we have the following proposition

Proposition 86. $\mathcal{H} \subset \overline{\mathcal{G}(2,1)}$.

Proof. The fact that

$$\dim(\mathcal{H} \backslash (\mathcal{G}(a,1) \cap \mathcal{H})) \leqslant 36. \tag{2.15}$$

is a very technical result that requires several intermediate steps, whose proof can be found in [2, Proposition 19]. Assuming it, it follows that $\mathcal{H}$ cannot fullfill an irreducible component of $\mathcal{B}(0,5)$. Since every locally free sheaf in $\mathcal{H}$ has $\alpha-$invariant equals to 1, we need to prove that if $E \in \mathcal{H}$ then $E \notin \mathcal{E}(3,2,0)$.

Now, suppose by contradiction that there exists a vector bundle $E \in \mathcal{H} \cap \overline{\mathcal{E}(3,2,0)}$. By the inferior semi-continuity of the dimension of the cohomology groups of coherent sheaves, one has that $h^1(E(-2)) \geqslant 3$. However, one can check from the display of the monad (2.14) that $\dim \mathrm{H}^1(E(-2)) = 1 < 3$. It follows that the family $\mathcal{H}$ must lie in $\overline{\mathcal{G}(2,1)}$. □

We finally have at hand all the ingredients needed to complete the proof of our second main result, namely the characterization of the irreducible components of $\mathcal{B}(0,5)$. We will proof the following result.

Theorem 87. The moduli space $\mathcal{B}(0,5)$ has exactly 3 irreducible components, namely:

(i) the **instanton component**, of dimension 37, which consists of those bundles given as cohomology of monads of the form

$$0 \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to 12 \cdot \mathcal{O}_{\mathbb{P}^3} \to 5 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \quad \text{and} \tag{2.16}$$

$$0 \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \to 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 3 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(2) \to 0; \tag{2.17}$$

(ii) the **Ein component**, of dimension 40, which consists of those bundles given as cohomology of monads of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0; \tag{2.18}$$

iii) the closure of the family $\mathcal{G}(2,1)$, of dimension 37, which consists of those bundles given as cohomology of monads of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \quad \text{and} \tag{2.19}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 6 \cdot \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to 2 \cdot \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{2.20}$$

Proof. Rao showed in [58] that the family of bundles obtained as cohomology of monads of the form (2.11) is irreducible, of dimension 36, and it lies in a unique component of $\mathcal{B}(0,5)$.

Since instanton bundles of charge 5, i.e. the cohomologies of monads of the form (2.10), yield an irreducible family of dimension 37, it follows that the set

$$\mathcal{I} := \{[E] \in \mathcal{B}(0,5) \mid \alpha(E) = 0\}$$

forms a single irreducible component of $\mathcal{B}(0,5)$, of dimension 37, whose generic point corresponds to an instanton bundle. In addition, every $[E] \in \mathcal{I}$ satisfies $\mathrm{H}^1(\mathrm{End}(E)) = 37$; this was originaly proved by Katsylo and Ottaviani for instanton bundles [39], and by Rao for the cohomologies of monads of the form (2.11) [58, Section 3]. Therefore, we also conclude that $\mathcal{I}$ is nonsingular.

Our next step is to analyse those bundles with Atiyah–Rees invariant equal to 1.

Hartshorne proved in [25, Theorem 9.9] that the family of stable rank 2 bundles E with $c_1(E) = 0$ and $c_2(E) = 5$ whose spectrum is $(-2,-1,0,1,2)$ form an irreducible, nonsigular family of dimension 40. Such bundles are precisely those given as cohomologies of monads of the form (2.12), cf. [27, Table 5.3, page 804], which is a particular case of a class of monads studied by Ein in [15]. From these references, we conclude that the closure of the family of vector bundle arising as cohomology of monads of the form (2.12) is an oversized irreducible component of $\mathcal{B}(5)$ of dimension 40.

We proved above that the bundles arising as cohomology of monads of the form (2.13) form a third irreducible component of dimension 37, while those bundles arising as cohomology of monads of the form (2.14), denoted by $\mathcal{H}$, form an irreducible family of dimension 36. It follows by the Proposition 86 that latter must lie in $\overline{\mathcal{G}(2,1)}$, concluding our proof.

□

For the sake of completeness, we summarize all the information in the theorem, and the discrete invariants of stable rank 2 bundles with $c_1 = 0$ and $c_2 = 5$ in the following table.

Table 1 – Irreducible components of $\mathcal{B}(0,5)$

Component	**Dimension**	**Monads**	**Spectra**	**α-invariant**
Instanton	37	(2.10)	(0,0,0,0,0)	0
		(2.11)	(-1,-1,0,1,1)	
Ein	40	(2.12)	(-2,-1,0,1,2)	1
Modified Instanton	37	(2.13)	(-1,0,0,0,1)	1
		(2.14)		

3 Moduli of Torsion Free Sheaves

The moduli space of rank 2 torsion free sheaves on $\mathbb{P}^3$ provides a natural compactification to the moduli space rank 2 locally free sheaves on $\mathbb{P}^3$, and although the study of its properties was a problem rised by Hartshorne in his problems list in late 70's, [24], it is suprising how little we know about them. Besides its intrinsecal interest, that is, in order to know more about locally free sheaves it makes sense to try to know more about torsion free sheaves. Moreover, the torsion free sheaves have, as predicted by Hartshorne, see [24, Problem 7], some significance to mathematical physics see [44].

In this chapter we compute the dimensions of the Ext groups of torsion free sheaves in terms of their Chern classes, and use it in order to produce the examples of irreducible components of the moduli space of torsion free sheaves. These results are then explored in order to prove that the number of irreducible components of $\mathcal{M}(c_1, c_2, 0)$ whose generic point correspond to a sheaf with 0-, 1-, or mixed dimensional singularities goes to infinity as c_2 goes to infinity, thus showing that the problem of computing the number of irreducible components of $\mathcal{M}(c_1, c_2, 0)$ becomes more complicated for higher valuers of c_2. Then, we study the moduli spaces $\mathcal{M}(-1, 2, c_3)$.

3.1 First Computations

In order to study the moduli spaces of torsion free sheaves on $\mathbb{P}^3$ we will need an explicit method to compute $\dim \operatorname{Ext}^1(E, E)$, which gives us the dimension of the tangent space of the isomorphism class of a stable torsion free sheaf E as a point the moduli space. Our main goal in this section is to prove the following theorem.

Theorem 88. Let E be a stable rank 2 torsion free sheaf on $\mathbb{P}^3$. Then

$$\dim \operatorname{Ext}^1(E, E) - \dim \operatorname{Ext}^2(E, E) = 8c_2(E) - 3 - c_1(E)^2.$$

Note that this result generalizes [36, Lemma 5d)] and [36, Lemma 10], which establish the formula above for stable rank 2 torsion free sheaves with 0- and 1-dimensional singularities, respectively, in the case $c_1(E) = 0$. The proofs for sheaves with 0- and 1-dimensional singularities with arbitrary c_1 are, mutatis mutandis, the ones in [36]; therefore, we only include here the proof for sheaves with mixed singularities.

Lemma 89. If E is a torsion free sheaf on $\mathbb{P}^3$, then

(i) $\mathrm{Ext}^1(E, E) = \mathrm{H}^1(\mathcal{H}om(E, E)) \oplus \ker d_2^{01}$;

(ii) $\mathrm{Ext}^2(E, E) = \ker d_3^{02} \oplus \ker d_2^{11} \oplus \operatorname{coker} d_2^{01}$;

(iii) $\mathrm{Ext}^3(E, E) = \operatorname{coker} d_3^{02}$.

Here, d_j^{pq} are the differentials in the j-th page of the spectral sequence for local to global ext's $E_2^{pq} := \mathrm{H}^p(\mathcal{E}xt^q(E, E))$. In particular, we have

$$\sum_{j=0}^{3}(-1)^j \dim \mathrm{Ext}^j(E, E) = \chi(\mathcal{H}om(E, E)) - \chi(\mathcal{E}xt^1(E, E)) + h^0(\mathcal{E}xt^2(E, E)).$$

Proof. The first part is a standard calculation with the spectral sequence $E_2^{pq} := \mathrm{H}^p(\mathcal{E}xt^q(E, E))$, which converges in its fourth page, since the spectral maps vanishes since $\mathcal{E}xt^q(E, E) = 0$ for $q \geqslant 4$ and $\mathrm{H}^p(\mathcal{E}xt^q(E, E))$ for $p \geqslant 4$. Note that $\mathrm{H}^p(\mathcal{E}xt^q(E, E)) = 0$ for $p \geqslant 2$ and $q \geqslant 1$, since $\dim \mathcal{E}xt^q(E, E) \leqslant 1$ for $q \geqslant 1$. Furthermore, applying the functor $\mathcal{H}om(\cdot, E)$ to the fundamental sequence (3.1):

$$0 \to E \to E^{\vee\vee} \to Q_E \to 0 \tag{3.1}$$

we get an epimorphism $\mathcal{E}xt^3(E^{\vee\vee}, E) \twoheadrightarrow \mathcal{E}xt^3(E, E)$ and the isomorphism $\mathcal{E}xt^2(E, E) \simeq \mathcal{E}xt^3(Q_E, E)$; however, the sheaf on the left vanishes because $E^{\vee\vee}$ is reflexive, and therefore has homological dimension equals to 1, so $\mathcal{E}xt^3(E, E) = 0$ as well. Finally, we also check that $\dim \mathcal{E}xt^2(E, E) = 0$, whenever nontrivial; indeed, E admits a resolution of the form

$$0 \to L_2 \to L_1 \to L_0 \to E \to 0, \tag{3.2}$$

where L_k are locally free sheaves; we then get an epimorphism

$$\mathcal{E}xt^3(Q_E, \mathcal{O}_{\mathbb{P}^3}) \otimes L_0 \twoheadrightarrow \mathcal{E}xt^3(Q_E, E),$$

which implies that $\dim \mathcal{E}xt^3(Q_E, E) = 0$ since $\dim \mathcal{E}xt^3(Q_E, \mathcal{O}_{\mathbb{P}^3}) = 0$.

The second claim is an immediate consequence of the first, together with $\dim \mathcal{E}xt^2(E, E) = 0$. □

Assuming that E is μ-semistable provides an useful simplification of the previous general result.

Lemma 90. If E be a μ-semistable torsion free sheaf on $\mathbb{P}^3$, then:

(i) $\mathrm{Ext}^1(E, E) = \mathrm{H}^1(\mathcal{H}om(E, E)) \oplus \ker d_2^{01}$;

(ii) $\mathrm{Ext}^2(E,E) = \mathrm{H}^0(\mathcal{E}xt^2(E,E)) \oplus \mathrm{H}^1(\mathcal{E}xt^1(E,E)) \oplus \operatorname{coker} d_2^{01}$;

(ii) $\mathrm{Ext}^3(E,E) = 0$.

Here, d_2^{01} and is the spectral sequence differential:

$$d_2^{01} : \mathrm{H}^0(\mathcal{E}xt^1(E,E)) \to \mathrm{H}^2(\mathcal{H}om(E,E)).$$

Proof. The last item is the easiest one: by Serre duality, we have

$$\mathrm{Ext}^3(E,E) \simeq \mathrm{Hom}(E, E(-4))^* = 0,$$

with the vanishing given by μ-semistability.

In addition, we argue that μ-semistability also implies that $\mathrm{H}^3(\mathcal{H}om(E,E)) = 0$. Indeed, applying the functors $\mathcal{H}om(-,E)$ and $\mathcal{H}om(E^{\vee\vee},-)$ to the fundamental sequences (3.1) we obtain, respectively,

$$0 \to \mathcal{H}om(E^{\vee\vee},E) \to \mathcal{H}om(E,E) \to \mathcal{E}xt^1(Q_E,E) \to \cdots$$

and

$$0 \to \mathcal{H}om(E^{\vee\vee},E) \to \mathcal{H}om(E^{\vee\vee},E^{\vee\vee}) \to \mathcal{H}om(E^{\vee\vee},Q_E) \to \cdots$$

In both sequences, the rightmost sheaf has dimension $\leqslant 1$, hence so does the cokernel of the leftmost monomorphism, and it follows that

$$\mathrm{H}^3(\mathcal{H}om(E,E)) \simeq \mathrm{H}^3(\mathcal{H}om(E^{\vee\vee},E)) \simeq \mathrm{H}^3(\mathcal{H}om(E^{\vee\vee},E^{\vee\vee})).$$

However

$$\mathrm{H}^3(\mathcal{H}om(E^{\vee\vee},E^{\vee\vee})) = \mathrm{Ext}^3(E^{\vee\vee},E^{\vee\vee}) \simeq \mathrm{Hom}(E^{\vee\vee},E^{\vee\vee}(-4))^{\vee} = 0;$$

the first equality follows from the spectral sequence for local to global ext's for $E^{\vee\vee}$, the isomorphism in the middles is given by Serre duality, and the vanishing is a consequence of the μ-semistability of $E^{\vee\vee}$.

It follows that $d_2^{pq} = 0$ except for d_2^{01}, while $d_3^{pq} = 0$ for every p and q. This means that E_2^{pq} converges in its third page, providing the desired result.

□

The following technical lemma will be helpful in our next argument.

Lemma 91. Let F be a torsion free sheaf. If E is a subsheaf of F for which the quotient sheaf $Z := F/E$ is 0-dimensional, then

$$\sum_{j=0}^{3}(-1)^j\chi(\mathcal{E}xt^j(Z,E)) + \sum_{j=0}^{3}(-1)^j\chi(\mathcal{E}xt^j(F,Z)) = 0. \tag{3.3}$$

Proof. Break a locally free resolution of E as in (3.2) into two short exact sequences

$$0 \to L_2 \to L_1 \to K \to 0 \text{ and } 0 \to K \to L_0 \to E \to 0.$$

Applying functor $\mathcal{H}om(Z, -)$ and passing to Euler characteristic on the first sequence, we have:

$$\chi(\mathcal{E}xt^2(Z, K)) - \chi(\mathcal{E}xt^3(Z, K)) = \chi(\mathcal{E}xt^3(Z, L_2)) - \chi(\mathcal{E}xt^3(Z, L_1)) = \tag{3.4}$$

$$= (\mathrm{rk}(L_2) - \mathrm{rk}(L_1))\chi(Z).$$

since $\chi(\mathcal{E}xt^3(Z, L_k)) = \chi(\mathcal{E}xt^3(Z, \mathcal{O}_{\mathbb{P}^3}) \otimes L_k) = \mathrm{rk}(L_k) \cdot \chi(Z)$. Now, applying the functor $\mathcal{H}om(Z, -)$ to the second exact sequence we obtain the isomorphism $\mathcal{E}xt^1(Z, E) \simeq \mathcal{E}xt^2(Z, K)$ and passing to the Euler characteristic we have

$$\chi(\mathcal{E}xt^2(Z, E)) - \chi(\mathcal{E}xt^3(Z, E)) = \chi(\mathcal{E}xt^3(Z, K)) - \chi(\mathcal{E}xt^3(Z, L_0))$$

Subtracting $\chi(\mathcal{E}xt^1(Z, E))$ from the left hand side and $\chi(\mathcal{E}xt^2(Z, K))$ from the right hand side, and then substituting for (3.4) we have:

$$\sum_{j=0}^{3}(-1)^j \chi(\mathcal{E}xt^j(Z, E)) = (\mathrm{rk}(L_1) - \mathrm{rk}(L_2) - \mathrm{rk}(L_0)) \cdot \chi(Z) = -\,\mathrm{rk}(E)\chi(Z). \tag{3.5}$$

Since $\dim \mathcal{E}xt^j(Z, E) = 0$, we have

$$\chi(\mathcal{E}xt^j(Z, E)) = h^0(\mathcal{E}xt^j(Z, E)) = \dim \mathrm{Ext}^j(Z, E) \overset{\mathrm{SD}}{=} \dim \mathrm{Ext}^{3-j}(E, Z) = \chi(\mathcal{E}xt^{3-j}(E, Z)),$$

where the supercript SD indicates the use of Serre duality. The formula (3.5) applied to the sheaf F then yields

$$\sum_{j=0}^{3}(-1)^j \chi(\mathcal{E}xt^j(F, Z)) = \mathrm{rk}(F)\chi(Z).$$

The fact that $\mathrm{rk}(F) = \mathrm{rk}(E)$ provides the desired identity.

□

Lemma 92. Let E be a rank 2 torsion free sheaf with mixed singularities. Then:

$$\sum_{j=0}^{3}(-1)^j \dim \mathrm{Ext}^j(E, E) = -8c_2(E) + 4 + 2c_1(E)^2.$$

Proof. Let $Z_E \hookrightarrow Q_E$ the maximal 0-dimensional subsheaf of Q_E, and set $T_E := Q_E/Z_E$ to be the pure 1-dimensional quotient; we assume that both Z_E and T_E are nontrivial. Let E' be the kernel of the composed epimorphism $E^{\vee\vee} \twoheadrightarrow Q_E \twoheadrightarrow T_E$; note that it also fits into the following short exact sequence

$$0 \to E \to E' \to Z_E \to 0. \tag{3.6}$$

Moreover, $c_1(E') = c_1(E)$ and $c_2(E') = c_2(E)$. In addition, $(E')^{\vee\vee} \simeq E^{\vee\vee}$, and $Q_{E'} \simeq T_E$, thus E' is a torsion free sheaf with 1-dimensional singularities. It follows that E' has homological dimension 1 (that is $\mathcal{E}xt^p(E', G) = 0$ for $p \geqslant 2$ and every coherent sheaf G), so the proof of [25, Proposition 3.4] also applies for E', and we conclude that

$$\sum_{j=0}^{3}(-1)^j \dim \operatorname{Ext}^j(E', E') = -8c_2(E) + 4 + 2c_1(E)^2.$$

Therefore, it is enough to prove that

$$\sum_{j=0}^{3}(-1)^j \dim \operatorname{Ext}^j(E, E) = \sum_{j=0}^{3}(-1)^j \dim \operatorname{Ext}^j(E', E'),$$

which, by Lemma 89 is the same as showing that

$$\chi(\mathcal{H}om(E, E)) - \chi(\mathcal{E}xt^1(E, E)) + h^0(\mathcal{E}xt^2(E, E)) = \chi(\mathcal{H}om(E', E')) - \chi(\mathcal{E}xt^1(E', E')).$$

To see this, applying the functor $\mathcal{H}om(E', -)$ to the sequence (3.6) we obtain:

$$\begin{aligned}&\chi(\mathcal{H}om(E', E)) - \chi(\mathcal{H}om(E', E')) + \chi(\mathcal{H}om(E', Z_E)) - \\ &\chi(\mathcal{E}xt^1(E', E)) + \chi(\mathcal{E}xt^1(E', E')) - \chi(\mathcal{E}xt^1(E', Z_E)) \\ &= 0.\end{aligned}$$

Next, applying the functor $\mathcal{H}om(-, E)$ to the sequence (3.6) we have

$$\begin{aligned}&\chi(\mathcal{H}om(E', E)) - \chi(\mathcal{H}om(E, E)) + \chi(\mathcal{E}xt^1(Z_E, E)) - \\ &\chi(\mathcal{E}xt^1(E', E)) + \chi(\mathcal{E}xt^1(E, E)) - \chi(\mathcal{E}xt^2(Z_E, E)) \\ &= 0\end{aligned}$$

Taking the difference between these last two equations we obtain

$$\begin{gathered}\chi(\mathcal{H}om(E', E')) - \chi(\mathcal{E}xt^1(E', E')) = \chi(\mathcal{H}om(E, E)) - \chi(\mathcal{E}xt^1(E, E)) + \\ -\chi(\mathcal{E}xt^1(Z_E, E)) + \chi(\mathcal{E}xt^2(Z_E, E)) + \chi(\mathcal{H}om(E', Z_E)) - \chi(\mathcal{E}xt^1(E', Z_E)) = \\ \chi(\mathcal{H}om(E, E)) - \chi(\mathcal{E}xt^1(E, E)) + \chi(\mathcal{E}xt^3(Z_E, E)),\end{gathered}$$

with the second equality following from applying the formula established in Lemma 91 to the sheaves E and E'. Applying the functor $\mathcal{H}om(-, E)$ to the sequences

$$0 \to E' \to E^{\vee\vee} \to T_E \to 0 \quad \text{and} \quad 0 \to Z_E \to Q_E \to T_E \to 0$$

we conclude that $\mathcal{E}xt^3(T_E, E) = 0$ and $\mathcal{E}xt^3(Q_E, E) \simeq \mathcal{E}xt^3(Z_E, E)$. We already noticed in the proof of Lemma 89 that $\mathcal{E}xt^3(Q_E, E) \simeq \mathcal{E}xt^2(E, E)$, thus $\chi(\mathcal{E}xt^3(Z_E, E)) = \chi(\mathcal{E}xt^2(E, E)) = h^0(\mathcal{E}xt^2(E, E))$, as desired.

□

Gathering the above results we are in position to prove the Theorem 88.

Proof of Theorem 88. By Lemma 92, it is enough to show that $\dim \mathrm{Hom}(E, E) = 1$ and $\dim \mathrm{Ext}^3(E, E) = 0$, but these follow easily from the stability of E (see proof of 90). □

The following proposition will be a technical tool that will help us to explicitly compute the dimension of $\mathrm{Ext}^1(E, E)$ for certain torsion free sheaves.

Proposition 93. Let F be a stable rank 2 reflexive sheaf on $\mathbb{P}^3$, with $\dim \mathrm{Ext}^2(F, F) = 0$. Let Z be a sheaf of finite length in $\mathbb{P}^3$, T a sheaf supported in a pure 1 dimensional subscheme $C \subset \mathbb{P}^3$, $Q := Z \oplus T$ such that Sing $F \cap \mathrm{Supp}\, Q = \varnothing$ and $\varphi : F \to Q$ be an epimorphism. In addition, let $E := \ker \varphi$. Then the following claims are true:

a) E is a stable rank 2 torsion free sheaf.

b) $c_1(E) = c_1(F)$ and $c_2(E) = c_2(F) + \deg C$.

c) We have that

$$\mathrm{Ext}^2(E, E) = \mathrm{H}^0(\mathcal{E}xt^3(Z, E)) \oplus \mathrm{Ext}^3(T, E). \tag{3.7}$$

Proof. The items a) and b) are straightforward. We will prove c). First we will show that the spectral sequence map

$$d_2^{01} : \mathrm{H}^0(\mathcal{E}xt^1(E, E)) \to \mathrm{H}^2(\mathcal{H}om(E, E))$$

is an epimorphism.

Consider the exact sequence:

$$0 \to E \to F \to Q \to 0, \tag{3.8}$$

applying the functor $\mathcal{H}om(F, -)$ in the sequence (3.8), once $\mathrm{coker}(\mathcal{H}om(F, E) \to \mathcal{H}om(F, F))$ is supported in dimension 1, we have

$$\mathrm{H}^2(\mathcal{H}om(F, E)) \to \mathrm{H}^2(\mathcal{H}om(F, F)) \to 0 \tag{3.9}$$

Next apply $\mathrm{Hom}(F, -)$ in the sequence (3.8), by hypothesis, $\mathrm{Ext}^2(F, F) = 0$, then we have

$$\mathrm{Ext}^1(F, Q) \to \mathrm{Ext}^2(F, E) \to 0$$

To see that $\mathrm{Ext}^1(F,Q)$ vanishes, note that $\mathrm{Ext}^p(F,Q)=0$ for $p=2,3$ because F is reflexive. $\mathcal{E}xt^1(F,Q)=0$ because Sing $F\cap$ Supp $Q=\varnothing$. In addition, $\mathrm{H}^p(\mathcal{H}om(F,Q))=0$ for $p=2,3$ because $\dim Q=1$. From the local to global spectral sequence, $\mathrm{Ext}^1(F,Q)=\mathrm{H}^1(\mathcal{H}om(F,Q))$ which vanishes by hypothesis. Therefore $d_2^{01}:\mathrm{H}^0(\mathcal{E}xt^1(F,E))\to\mathrm{H}^2(\mathcal{H}om(F,E))$ is surjective. Then we have

$$\begin{tikzcd} \mathrm{H}^0(\mathcal{E}xt^1(F,E)) \arrow[r, "d_2^{01}"] \arrow[d] & \mathrm{H}^2(\mathcal{H}om(F,E)) \arrow[d] \\ \mathrm{H}^0(\mathcal{E}xt^1(E,E)) \arrow[r, "d_2^{01}"] & \mathrm{H}^2(\mathcal{H}om(E,E)), \end{tikzcd} \tag{3.10}$$

where the vertical arrow in the left is the natural map coming from the exact sequence (3.8), and horizontal maps came from the spectral sequence. Since the top row map, and the right vertical map are surjective, we have that the bottom map is surjective as we wanted. Now, applying $\mathcal{H}om(-,E)$ in the sequence (3.8) we have

$$\mathcal{E}xt^2(E,E)\simeq\mathcal{E}xt^3(Q,E)\simeq\mathcal{E}xt^3(Z,E)\oplus\mathcal{E}xt^3(T,E)$$

Furthermore,

$$\mathcal{E}xt^1(F,E)\longrightarrow\mathcal{E}xt^1(E,E)\xrightarrow{f}\mathcal{E}xt^2(Q,E)\longrightarrow 0,$$

note that $\dim\ker f=0$, since $\dim\mathcal{E}xt^1(F,E)=0$, thus

$$\mathrm{Ext}^2(E,E)=\mathrm{H}^0(\mathcal{E}xt^3(Z,E))\oplus\mathrm{H}^0(\mathcal{E}xt^3(T,E))\oplus\mathrm{H}^1(\mathcal{E}xt^2(T,E)). \tag{3.11}$$

Since F is reflexive, from [31, Proposition 1.1.6], we have $\mathcal{E}xt^p(T,F)=0$ for $p=0,1$ and codim Supp $\mathcal{E}xt^p(T,F)\geqslant q$ for $q=2,3$, once F is locally free along the support of fo T. Applying the functor $\mathcal{H}om(T,-)$ in the sequence (3.8) one sees that $\dim\mathcal{E}xt^p(T,E)=1$ for $p=1,2$, $\dim\mathcal{E}xt^3(T,E)=0$ and $\mathcal{H}om(T,E)=0$, using these facts, the spectral sequence for $\mathrm{Ext}^3(T,E)$ gives

$$\mathrm{Ext}^3(T,E)=\mathrm{H}^0(\mathcal{E}xt^3(T,E))\oplus\mathrm{H}^1(\mathcal{E}xt^2(T,E)). \tag{3.12}$$

Putting together the equations (3.11) and (3.12) we obtain the formula (3.7).

□

An important ingredient of the Proposition 93 is a family of stable reflexive sheaves, that fills out an irreducible component of the moduli space, with the expected dimension. A priori, it is not clear why such family should exists. In [36] the authors proved that indeed such families exists for infinitely many values of the second Chern class, provided that the first Chern class is even. Next, we state a theorem that shows that this happens also for sheaves with odd first Chern class.

Theorem 94. For each triple (a, b, c) of positive integers such that $3a + 2b + c$ is nonzero and odd, let $G_{(a,b,c)} := a \cdot \mathcal{O}_{\mathbb{P}^3}(-3) \oplus b \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus c \cdot \mathcal{O}_{\mathbb{P}^3}(-1)$. The family of rank 2 reflexive sheaves F obtained as the cokernel of the maps α below, whose degeneracy locus is is 0-dimensional

$$0 \to a \cdot \mathcal{O}_{\mathbb{P}^3}(-3) \oplus b \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus c \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (a + b + c + 2) \cdot \mathcal{O}_{\mathbb{P}^3} \to F(k) \to 0$$

fills out an irreducible, nonsingular, component $\mathcal{S}(a, b, c)$ of $\mathcal{R}(-1; n; m)$ of expected dimension $8n - 5$, where $k := (3a + 2b + c)/2$, so that $c_1(F) = -1$, n and m are given by the expressions:

$$n = \frac{1}{4}(3a + 2b + c + 1)^2 + \frac{6a}{2} + b,$$

$$= m(a, b, c) = 27\binom{a+2}{3} + 8\binom{b+2}{3} + \binom{c+2}{3} + 3(3a + 2b + 5)ab+ \tag{3.13}$$

$$+\frac{3}{2}(3a + c + 4)ac + (2b + 3c + 3)bc + 6abc$$

more precisely, $\widetilde{\mathcal{S}}(a, b, c) \subset \mathrm{Hom}\left(G_{(a,b,c)}, (a + b + c + 2) \cdot \mathcal{O}_{\mathbb{P}^3}\right)$ be the open subset consisting of monomorphisms with 0-dimensional degeneracy loci; then

$$\mathcal{S}(a, b, c) = \widetilde{\mathcal{S}}(a, b, c)/(\mathrm{Aut}(G_{(a,b,c)}) \times GL(a + b + c + 2))/\mathbb{C}^*).$$

Proof. Let $a, b, c \in \mathbb{Z}$, such that $3a + 2b + c$ is odd and non zero and, and consider the morphisms

$$\alpha : a \cdot \mathcal{O}_{\mathbb{P}^3}(-3) \oplus b \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus c \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to (a + b + c + 2) \cdot \mathcal{O}_{\mathbb{P}^3}$$

if the degeneracy locus $\Delta(\alpha) := \{x \in \mathbb{P}^3 \mid \alpha(x) \text{ is not injective}\}$ is 0-dimensional then the cokernel of α is a rank 2 reflexive sheaf on $\mathbb{P}^3$, which we normalize as to fit into the short exact sequence:

$$0 \to a \cdot \mathcal{O}_{\mathbb{P}^3}(-3) \oplus b \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus c \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (a + b + c + 2) \cdot \mathcal{O}_{\mathbb{P}^3} \to F(k) \to 0 \tag{3.14}$$

we set $k := (3a + 2b + c + 1)/2$, so that $c_1(F) = -1$.

For simplicity of notation, let $G_{(a,b,c)} := a \cdot \mathcal{O}_{\mathbb{P}^3}(-3) \oplus b \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus c \cdot \mathcal{O}_{\mathbb{P}^3}(-1)$.

The dimension of the family of rank 2 reflexive sheaves constructed as in the short exact sequence (3.14) is given by

$$\dim \operatorname{Hom}\left(G_{(a,b,c)}, (a+b+c+2)\cdot \mathcal{O}_{\mathbb{P}^3}\right) - \dim \operatorname{Aut}\left(G_{(a,b,c)}\right) - (a+b+c+2)^2 + 1 =$$

$$8k^2 + 24a + 8b - 5 = 8c_2(F) - 5$$

Now, note that for every F given by (3.14), we have that $\mathrm{H}^0(F(-1)) = 0$ thus F is always stable, hence, we only need to check that $\dim \operatorname{Ext}^2(F,F) = 0$, in order to have $\dim \operatorname{Ext}^1(F,F) = 8c_2(F) - 5$, but this follows applying the functor $\operatorname{Hom}(\cdot, F(k))$ to the sequence (3.14), and observing that $\mathrm{H}^1(F(t)) = 0$ for every $t \in \mathbb{Z}$ and that $\mathrm{H}^2(F(k)) = 0$. Therefore the family of sheaves given by (3.14) provides a component of the moduli space of stable rank 2 reflexive sheaves on $\mathbb{P}^3$ □

A case that deserves special attention is the is the case $a = b = 0$ and $c = 1$, that give us $c_1(F) = -1$, $c_2(F) = c_3(F) = 1$. In [25, Lemma 9.4] and [25, Lemma 9.4] is shown that every reflexive sheaf in $\mathcal{R}(-1,1,1)$ admits a resolution of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} 3\cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to F \to 0 \tag{3.15}$$

From this sequence we can obtain the splitting behaviour of a sheaf $F \in \mathcal{R}(-1,1,1)$. Indeed, each one of the 3 rows of the map α can be viewed as the equation of a hyperplane in $\mathbb{P}^3$, since α is injective, the hyperplane must intersect in exactly one point p, that coincides with the singularity of the sheaf F. Thus, if $l \subset \mathbb{P}^3$ is a line, if $p \notin l$, then the restriction of F on l, $F|_l$, is isomorphic to $\mathcal{O}_l(-1) \oplus \mathcal{O}_l$. On the other hand, if $p \in l$, from sequence (3.15), we have that $F|_l \simeq \mathcal{O}_p \oplus 2\mathcal{O}_l(-1)$. Summarizing, we have:

$$F|_l = \begin{cases} \mathcal{O}_l(-1) \oplus \mathcal{O}_l, & \text{if } p \notin l \\ \mathcal{O}_p \oplus 2\mathcal{O}_l(-1), & \text{if } p \in l \end{cases} \tag{3.16}$$

Moreover the [25, Theorem 9.3], shows that $\mathcal{R}(-1,1,1)$ is irreducible non-singular and rational of dimension 3. Then Theorem 94, implies that $\widetilde{\mathcal{S}}(0,0,1) = \mathcal{R}(-1,1,1)$. These sheaves will play an important role in the proof of the main results of this work.

3.2 Sheaves with mixed singularities

In [36] the authors produced examples of irreducible components with 0-dimensional singularities and pure 1-dimensional singularities, in the moduli space of rank 2 stable torsion free sheaves with first Chern class equals to 0, and in [32] the authors proved the existence of irreducible components in $\mathcal{M}(0,3,0)$ whose generic point is a sheaf with mixed singularities. The first natural question that arises is that if similar constructions can be made for sheaves with odd first Chern class, and if it is possible similar irreducible components for non zero third Chern class.

We will explicit construct examples of irreducible components of the moduli space of torsion free sheaves with mixed singularities. We refer the reader to [32] for some examples in $\mathcal{M}(0,3,0)$.

For the rest of this work, let $e \in \{-1,0\}$, and n and m be two integers such that $en \equiv m(\mathrm{Mod}\ 2)$.

Let $\mathcal{R}^*(e,n,m) \subseteq \mathcal{R}(e,n,m)$ be an irreducible component of the moduli space of stable reflexive sheaves with the expected dimension, and such that, for each $F \in \mathcal{R}^*(e,n,m)$, we have that $\mathrm{Ext}^2(F,F) = 0$. Recall that by Theorem 94 it always possible to find such components for $e = -1$. For the case $e = 0$, the same is also true due to [36, Theorem 8].

Let $\mathcal{S} \subset \mathrm{Sym}^s\ (\mathbb{P}^3)$ be the open dense subset of disjoint unions of s distinct points in $\mathbb{P}^3$. For any closed point $[F] \in \mathcal{R}^*(e,n,m)$, we can define the set

$$\mathcal{X}_{[F]} := \{l \times S \in \mathrm{G}(2,4) \times \mathcal{S} \ \mid\ l \cap S = \varnothing \ \text{ and } \ F|_l = \mathcal{O}_l(e) \oplus \mathcal{O}_l\}$$

For a pair $(l,S) \in \mathcal{X}_{[F]}$ let $r \geqslant e$, and $Q_{(l,S),r} := \mathcal{O}_S \oplus (i_*(\mathcal{O}_l))(r)$ where $i : l \hookrightarrow \mathbb{P}^3$ is a closed immersion. Consider the open dense subset $\mathrm{Hom}(F, Q_{(l,S),r})_e \subset \mathrm{Hom}(F, Q_{(l,S),r})$ of the epimorphisms $F \to Q_{(l,S),r}$. For any element $\phi \in \mathrm{Hom}(F, Q_{(l,S),r})_e$ the torsion free sheaf $E_\phi := \ker \phi$ is stable, and defines a closed point in $\mathcal{M}(e, n+1, m-2r-2s-2-e)$. Furthermore, $E_\phi \simeq E_{\phi'}$ if, and only if, there is a $g \in \mathrm{Aut}(Q_{(l,S),r})$ such that $\phi = g \circ \phi'$. Denote by $[\phi]$ the equivalence class of ϕ modulo $\mathrm{Aut}(Q_{(l,S),r})$. Now consider the set

$$\begin{aligned}\tilde{\mathcal{X}}(e,n,m,r,s) := \{x = ([F],(l,S),[\phi_x])| \ \ [F] \in \mathcal{R}(c_1,n,m) \ \ (l,S) \in \mathcal{X}_{[F]},\\ \text{and } [\phi_x] \in \mathrm{Hom}(F, Q_{(l,S),r})_e/\mathrm{Aut}(Q_{(l,S),r})\}\end{aligned}$$

By hypothesis, $\mathcal{R}^*(e,n,m)$ is a reduced and irreducible scheme, hence $\tilde{\mathcal{X}}(e,n,m,r,s)$ is a reduced irreducible scheme. To see this, consider the product $\mathcal{R}(e,n,m) \times (\mathbb{P}^3)^s \times G(2,4)$ and the subset

$$\begin{aligned}\left(\mathcal{R}^*(e,n,m) \times (\mathbb{P}^3)^s \times G(2,4)\right)^0 := \{([F], q_1, \ldots, q_s, l) \mid q_i \neq q_j \ ,\ q_i \notin \mathrm{Sing}(F),\\ q_i \notin l, \ \text{and}, \mathrm{Sing}(F) \cap l = \varnothing\}\end{aligned}$$

By Grauert-Mulich Theorem, we get that $\left(\mathcal{R}^*(e,n,m)\times(\mathbb{P}^3)^s\times G(2,4)\right)^0$ is an open dense subset of $\mathcal{R}(e,n,m)\times(\mathbb{P}^3)^s\times G(2,4)$. Note that one has the surjective projection

$$\tilde{\mathcal{X}}(e,n,m,r,s)\twoheadrightarrow\left(\mathcal{R}^*(e,n,m)\times(\mathbb{P}^3)^s\times G(2,4)\right)^0\quad,\quad([F],Q_{(l,s),r},\ \varphi)\mapsto([F],Q_{(l,s),r})$$

onto an irreducible base variety of dimension $8n+3s+2e+1$, with irreducible fibers given by

$$\mathrm{Hom}(F,Q_{(l,s),r})_e/\mathrm{Aut}(Q_{(l,s),r})\overset{\text{open}}{\hookrightarrow}\mathrm{Hom}(F,Q_{(l,s),r})/\mathrm{Aut}(Q_{(l,s),r})$$

which have dimension $2s+2r-e+2-s-1=s+2r-e+1$.

Note that since $F\cap\operatorname{supp}Q_{(l,s),r})=\varnothing$, there exists an open set $U\subset\mathbb{P}^3$ such that U contains l and $\mathcal{S}$, and F trivializes on U. Then it follows that $\mathrm{Hom}(F,Q_{(l,s),r})\simeq\mathrm{Hom}(2\mathcal{O}_U,i^*\mathcal{O}_l(r))\bigoplus_{i=1}^{s}\mathrm{Hom}(2\mathcal{O}_U,\mathcal{O}_{p_i})$ and therefore:

$$\mathrm{Hom}(F,Q_{(l,s),r})/\mathrm{Aut}(Q_{(l,s),r})\simeq\mathbb{P}(\mathrm{Hom}(2\mathcal{O}_U,i^*\mathcal{O}_l(r)))\times_{i=1}^{s}\mathbb{P}(\mathrm{Hom}(2\mathcal{O}_U,\mathcal{O}_{p_i}))$$

Therefore $\mathrm{Hom}(F,Q_{(l,s),r})/\mathrm{Aut}(Q_{(l,s),r})$ is a Segre Variety thus it is irreducible.

Now, To each $\mathbf{t}:=([F],Q_{(l,s),r},\varphi)\in\tilde{\mathcal{X}}(e,n,m,r,s)$ one associates the sheaf $E(\mathbf{t}):=\ker\{\varphi:F\twoheadrightarrow Q_{(l,s),r}\}$ which defines a point $[E(\mathbf{t})]$ in $\mathcal{M}(e,n+1,m+2+e-2r-2s)$, this map defines a natural injective morphism

$$\Psi:\tilde{\mathcal{X}}(e,n,m,r,s)\hookrightarrow\mathcal{M}(e,n+1,m+2+e-2r-2s)$$

Let $\mathcal{X}(e,n,m,r,s):=\Psi(\tilde{\mathcal{X}}(e,n,m,r,s))$. The dimension of the scheme $\tilde{\mathcal{X}}(e,n,m,r,s)$ (and consequently of the scheme $\mathcal{X}(e,n,m,r,s)$ can be computed in the following way

$$\dim\tilde{\mathcal{X}}(e,n,m,r,s)=\dim\mathcal{R}(e,n,m)+\dim\mathrm{G}(2,4)+\dim S+\dim\mathrm{Hom}(F,Q_{(l,S),r})_e/\mathrm{Aut}(Q_{(l,S),r})$$

from what follows

$$\dim\tilde{\mathcal{X}}(e,n,m,r,s)=8n+4s+2r+2+e\tag{3.17}$$

We are going to prove that, for each point $E_\varphi\in\mathcal{X}(e,n,m,r,s)$, we have that $\dim\mathrm{Ext}^1(E_\phi,E_\phi)=\dim\tilde{\mathcal{X}}(e,n,m,r,s)$.

Theorem 95. Given positive integers n,m such that exists an irreducible component $\mathcal{R}^*(e,n,m)\subset\mathcal{R}(e,n,m)$ with the expected dimension, and such that, for each $F\in\mathcal{R}(e,n,m)$, $\mathrm{Ext}^2(F,F)=0$, we have that every $E_\phi\in\mathcal{X}(e,n,m,r,s)$, as defined above, satisfies

$$\dim\mathrm{Ext}^1(E_\phi,E_\phi)=\begin{cases}8n+4s+2r+2+e, & \text{if } r\geqslant 2\\ 8n+4s+3, & \text{otherwise}\end{cases}\tag{3.18}$$

In particular, for each $r \geqslant 2$, and s such that $0 \leqslant s \leqslant 2r + 2 + e - m$, or, for $r = 1$, $s = 0$, and $n = m = 1$, we have that every $E_\phi \in \mathcal{X}(e, n, m, r, s)$ satisfies

$$\dim \operatorname{Ext}^1(E_\phi, E_\phi) = \dim \mathcal{X}(e, n, m, r, s) \tag{3.19}$$

Hence, $\mathcal{X}(e, n, m, r, s)$ is an open dense subset of an irreducible component $\mathrm{X}(e, n, m, r, s) := \overline{\mathcal{X}(e, n, m, r, s)}$ of $\mathcal{M}(e, n+1, m+2+e-2r-2s)$ of dimension $8n + 4s + 2r + 2 + e$.

Proof. Consider an $E_\phi \in \mathcal{X}(e, n, m, r, s)$. By definition we have the sequence

$$0 \to E_\phi \to F \to Q_{(l,S),r} \to 0 \tag{3.20}$$

Fom Proposition 93, we just need to compute $\dim \mathrm{H}^0(\mathcal{E}xt^3(\mathcal{O}_S, E_\phi)) + \dim \operatorname{Ext}^3(i_*\mathcal{O}_l(r), E_\phi)$. As in [36, Proposition 6] one can check that $\dim \mathrm{H}^0(\mathcal{E}xt^3(\mathcal{O}_S, E_\phi)) = 4s$. We will include the proof here for completeness.

If $p \notin \mathcal{S}$, then clearly $\mathcal{E}xt^3(\mathcal{O}_S, E_\phi) = 0$.

Take $p \in \mathcal{S}$; restricting the sequence (3.8) to an open affine subset U of $\mathbb{P}^3$ containing p but none of the other singularities of F, we have the following short exact sequence of sheaves on U:

$$0 \to \mathcal{O}_U \oplus I_{p/U} \to 2 \cdot \mathcal{O}_U \to \mathcal{O}_{p/U} \to 0 \tag{3.21}$$

where $I_{p/U}$ denotes the ideal sheaf of the point $p \in U$ and $\mathcal{O}_{p/U}$ denotes the structure sheaf of the point p as a subscheme of U. Therefore we have that :

$$H^0(\mathcal{E}xt^3(\mathcal{O}_{p/U}, E_\phi|_U)) = H^0(\mathcal{E}xt^3_{\mathcal{O}_U}(\mathcal{O}_{p/U}, \mathcal{O}_U)) \oplus H^0(\mathcal{E}xt^3_{\mathcal{O}_U}(\mathcal{O}_{p/U}, I_{p/U})). \tag{3.22}$$

Now, applying functor $\mathcal{H}om(-, \mathcal{O}_{p,\mathbb{P}^3})$ in the sequence (3.23), one sees that

$\mathcal{E}xt^3_{\mathcal{O}_U}(\mathcal{O}_{p/U}, I_{p/U}) \simeq \mathcal{E}xt^2_{\mathcal{O}_U}(I_{p/U}, I_{p/U}))$. Bu, by the proof of [36, Proposition 6] one sees that the later has length 3. Therefore, for each $p = q_j \in \operatorname{Supp}(Q)$, we have that $h^0(\mathcal{E}xt^2_{\mathcal{O}_U}(I_{p/U}, I_{p/U})) = 3$.

Finally, since $\mathcal{E}xt^3(\mathcal{O}_{p/\mathbb{P}^3}, \mathcal{O}_U) \simeq \mathcal{O}_{p/\mathbb{P}^3}$, it has length 1 and them $h^0(\mathcal{E}xt^2_{\mathcal{O}_U}(I_{p/U}, \mathcal{O}_{p/U})) = 1$, for each $p = q_j \in \mathcal{S}$. Therefore, each of the s points contributes with 4 for the dimension of $\dim \mathcal{E}xt^3(\mathcal{O}_S, E_\phi)$, hence $\dim \mathrm{H}^0(\mathcal{E}xt^3(\mathcal{O}_S, E_\phi)) = 4s$

Now, $\operatorname{Ext}^3(i_*\mathcal{O}_l(r), E_\phi) \simeq \operatorname{Hom}(E_\phi, i_*\mathcal{O}_l(r-4))^\vee$ by Serre Duality. And $\operatorname{Hom}(E_\phi, i_*\mathcal{O}_l(r-4)) \simeq \mathrm{H}^0(\mathcal{H}om(E_\phi, i_*\mathcal{O}_l(r-4)))$ by local to global spectral sequence.

To compute this last cohomology group, we need to understand the restriction of E_ϕ in the line l. Applying the functor $i^*(-\otimes i_*\mathcal{O}_l)$ we have

$$0 \longrightarrow i^*Tor_1(i_*\mathcal{O}_l(r), i_*\mathcal{O}_l) \longrightarrow E_\phi|_l \xrightarrow{f} \mathcal{O}_l(e)\oplus\mathcal{O}_l \xrightarrow{g} \mathcal{O}_l(r) \longrightarrow 0 \tag{3.23}$$

Since $i^*Tor_1(i_*\mathcal{O}_l(r), i_*\mathcal{O}_l) \simeq N_l^\vee \otimes \mathcal{O}_l(r) \simeq 2.\mathcal{O}_l(r-1)$

from the sequence (3.23) we see that $\ker g \simeq \mathcal{O}_l(e-r)$. From what follows that $E_\phi|_l \simeq 2.\mathcal{O}_l(r-1)\oplus\mathcal{O}_l(e-r)$. Now, $\mathcal{H}om(E_\phi, i_*\mathcal{O}_l(r-4)) = 2.\mathcal{O}_l(-3)\oplus\mathcal{O}_l(2r-e-4)$. Therefore, $\dim \mathrm{H}^0(\mathcal{H}om(E_\phi, i_*\mathcal{O}_l(r-4))) = 2r-e-3$ if $r \geqslant 2$ and 0 otherwise. Therefore if $r \geqslant 2$ we have

$$\dim \mathrm{Ext}^1(E_\phi, E_\phi) = 8n + 4s + 2r + 2 + e$$

and

$$\dim \mathrm{Ext}^1(E_\phi, E_\phi) = 8n + 4s + 3$$

otherwise.

This means that if $r \geqslant 2$, we will have $\dim \mathrm{Ext}^1(E_\phi, E_\phi) = \tilde{\mathcal{X}}(e,n,m,r,s)$, and if $r=-1$, $s=0$ and $n=m=1$ we will have $\dim \mathrm{Ext}^1(E_\phi, E_\phi) = \tilde{\mathcal{X}}(e,n,m,r,s) = 11$, from where follows our result. $\square$

Note that the examples of irreducible components of sheaves with 0-dimensional singularities described in [36] can be obtained as a particular case of the Theorem 95, by ommiting the line l in the construction of the family and repeating the arguments of the proof, mutatis mutandis. Once this includes sheaves with odd determinant, we will state this result in the next theorem.

Theorem 96. For every nonsingular irreducible component $\mathcal{R}^*(e,n,m)$ of $\mathcal{R}(e,n,m)$ of expected dimension $8n-3+2e$, there exists an irreducible component $\mathrm{T}(e,n,m,l)$ of dimension $8n-3+2e+4l$ in $\mathcal{M}(e,n,m-2l)$ whose generic sheaf $[E]$ satisfies $[E^{\vee\vee}] \in \mathcal{R}^*(e,n,m)$ and $\mathrm{lenght}(Q_E) = l$.

Additionaly, it is important to highlight that $\mathrm{X}(e,n,m,r,0)$ is an irreducible component of $\mathcal{M}(e, n+1, m+2+c_1-2s)$ whose generic sheaf corresponds to a sheaf with pure 1-dimensional singularities different from those described in [36, Theorem 15].

3.3 Ein Type Results

Our goal in this section is to prove that each one of the irreducible components described in the previous paragraphs grow in number as the second Chern class of the sheaves grows. Indeed we will prove:

Theorem 97. The number η_n of irreducible components of $\mathcal{M}(-1, n, 0)$ which the generic point corresponds to a sheaf with mixed singularities goes to infinite as n goes to infinite.

Proof. For any odd integer $q \geqslant 1$ set $n_q = 9q^2 - 6q - 1$ and $i,\ 0 \leqslant i \leqslant q-1$, let $a_{q,i} = i,\ b_{q,i} = 3q-3i-3,\ c_{q,i} = 3i+2$. Then, according to Theorem 8 of [36] the sheaf $F \in S(a_{q,i}, b_{q,i}, c_{q,i})$ belongs to an irreducible component $S(a_{q,i}, b_{q,i}, c_{q,i})$ of $\mathcal{R}(-1; n_q, m_{q,i})$, where $m_{q,i} = m(a_{q,i}, b_{q,i}, c_{q,i})$ is an odd integer given by (3.13). Now, by Theorem 95, for each integer $s \leqslant n_q - 1$, and $2r = m_{q,i} + 1 - 2s$ corresponds to a component $\mathrm{X}(-1, n-1, m_{q,i}, r, s) \subseteq \mathcal{M}(-1, n_q, 0)$ of dimension $8n_q + 4s + 2(n_q + 1 - s) + 1$ For each odd q we obtain q different families of stable reflexive sheaves with the expected dimension, and each such family corresponds to $n_q - 1$ families of torsion free sheaves such that the generic point is a sheaf with mixed singularities. Therefore for each q we have at least $9q^3 - 6q^2 - q$ irreducible components of $\mathcal{M}(-1, n_q, 0)$ whose generic point is a stable torsion free sheaf with mixed singularities. □

As discussed in the end of the previous section, from the irreducible components with mixed singularities, we can obtain irreducible components with 0−dimensional singularities and pure 1-dimensional singularities, from this observation, and Theorem 97, we can easily obtain the following Corollary.

Corollary 98. Let ζ_n and ξ_n denote the number of irreducible components of $\mathcal{M}(-1, n, 0)$ whose generic points correspond to sheaves with 0-dimensional singularities and with 1-dimensional singularities, respectively. Then

$$\limsup_{n\to\infty} \zeta_n = \infty, \text{ and, } \limsup_{n\to\infty} \xi_n = \infty$$

.

Proof. It follows from the proof of Theorem 97, considering $s = 0$, for the pure 1-dimensional singularities, and ommiting the line in the 0-dimensinal case. □

3.4 Irreducible components of $\mathcal{M}(-1, 2, c_3)$

In the previous sections our results ensured the existence of irreducible components of the moduli spaces of torsion free sheaves with prescribed singularities, but for given chern classes we were not able to describe all irreducible components of the moduli space. The aim of this section is to consider this problem for small values of c_2, in order to obtain to have the complete characterization of the moduli spaces, and to illustrate why this study becomes too complicated for large values of c_2.

More precisely, in this section we will describe the irreducible components of the moduli spaces $\mathcal{M}(-1, 2, c_3)$, for $c_3 = 0, 2, 4$.

For the convenience of the reader, in the following Proposition we will fix some numerical invariants of torsion free sheaves that we will use in this entire section.

Proposition 99. Let E be a normalized stable torsion free sheaf, $E^{\vee\vee}$ its double dual and $Q_E := E^{\vee\vee}/E$. The following holds:

a) $c_1(E^{\vee\vee}) = c_1(E)$

b) $c_2(E^{\vee\vee}) = c_2(E) - \operatorname{mult} Q_E$, where $\operatorname{mult} Q_E$ is the multiplicity of the sheaf Q_E.

c) $c_3(E^{\vee\vee}) = (c_3(E) + c_3(Q_E)) - c_1(E) \operatorname{mult} Q_E$

Additionally, it follows that $c_2(E) \geqslant \operatorname{mult} Q_E$, from the stability of E.

Proof. Since E is torsion free, it fits in the following exact sequence:

$$0 \to E \to E^{\vee\vee} \to Q_E \to 0 \tag{3.24}$$

Computing the Chern classes we have the item *a*) to *c*), and since E is stable, $E^{\vee\vee}$ is stable, then $c_2(E^{\vee\vee}) \geqslant 0$. Then by 99 b), $0 \leqslant \deg Q_E \leqslant c_2(E)$. □

The next Lemma is an easy technical result that we use later in this section.

Lemma 100. For each $F \in \mathcal{R}(-1, 1, 1)$, consider the set $Y_F := \{l \in G(2, 4);\ \operatorname{Sing} F \subset l\}$, and the set:

$$Y(r) := \{(F, l, \varphi);\ (F, l) \in \mathcal{R}(-1, 1, 1) \times Y_F;\ \varphi \in \operatorname{Hom}(F, i_*\mathcal{O}_l(r))_e/\operatorname{Aut}(F, i_*\mathcal{O}_l(r)\}$$

Then, for each $r \in \{-1, 0, 1\}$, the set $Y(r)$ is an irreducible scheme of dimension $8 + 2r$. Additionally, the image of the morphism $Y(r) \to \mathcal{M}(-1, 2, 2-2r)$ that, for each triple $(F, l, \varphi) \in$

$Y(r)$ associates the sheaf $[\ker\varphi] \in \mathcal{M}(-1,2,2-2r)$ never fulfills an irreducible component of $\mathcal{M}(-1,2,2-2r)$.

Proof. For each $F \in \mathcal{R}(-1,1,1)$, $\operatorname{Sing} F$ is an unique point, this means that the set Y_F of lines $l \in G(2,4)$ such that $\operatorname{Sing} F \subset l$ is a surface in the Grassmannian isomorphic to $\mathbb{P}^2$. Therefore it is irreducible of dimension 2. To see that the dimension of $\operatorname{Hom}(F, i_*\mathcal{O}_l(r)_e/\operatorname{Aut}(F, i_*\mathcal{O}_l(r)\}$ is $3+2r$, apply functor $\mathcal{H}om(-, i_*\mathcal{O}_l(r))$ in the sequence (3.15), and recall that $\dim \mathrm{H}^0(\mathcal{E}xt^1(F, i_*\mathcal{O}_l(r))) = 1$.

Putting all these data together, we define the set

$$Y(r) := \{(F,l,\varphi);\ (F,l) \in \mathcal{R}(-1,1,1) \times Y_F,\ \varphi \in \operatorname{Hom}(F, i_*\mathcal{O}_l(r))_e/\operatorname{Aut}(F, i_*\mathcal{O}_l(r)\}$$

which by construction is an irreducible scheme of dimension $8+2r$. Indeed one has the surjective projection

$$Y(r) \twoheadrightarrow \mathcal{R}(-1,1,1) \times Y_F \quad , \quad ([F], l, \varphi) \mapsto ([F], l)$$

onto an irreducible scheme of dimension 5, with fibers given by

$$\operatorname{Hom}(F, i_*\mathcal{O}_l(r))_e/\operatorname{Aut}(i_*\mathcal{O}_l(r)) \overset{\text{open}}{\hookrightarrow} \operatorname{Hom}(F, i_*\mathcal{O}_l(r))/\operatorname{Aut}(i_*\mathcal{O}_l(r))$$

which have dimension $3+2r$. □

With the previous Lemma, we are already in position to prove the first main result of this section.

Theorem 101. The moduli space of rank 2 stable sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = -1, c_2 = 2, c_3 = 4$, is the closure of the moduli space of the rank 2 reflexive sheaves with Chern classes $c_1 = -1, c_2 = 2, c_3 = 4$, hence, it is irreducible, generically smooth, of dimension 11.

Proof. By [25, Thm 9.2], $\mathcal{R}(-1,2,4)$ is irreducible of dimension 11, and, $\overline{\mathcal{R}(-1,2,4)}$ is an irreducible component of $\mathcal{M}(-1,2,4)$.

Consider $E \in \mathcal{M}(-1,2,4) \backslash \mathcal{R}(-1,2,4)$. By Proposition 99, $0 \leqslant \operatorname{mult} Q_E \leqslant 2$. We will study the possibilities for $\operatorname{mult} Q_E$.

i) If $\operatorname{mult} Q_E = 2$, then $c_2(E^{\vee\vee}) = 0$, and by [25], $c_3(E^{\vee\vee}) \leqslant c_2(E^{\vee\vee})^2$, therefore $c_3(E^{\vee\vee}) = 0$. Hence $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}$, which contradicts the stability of $E^{\vee\vee}$.

ii) If $\operatorname{mult} Q_E = 1$, then $c_2(E^{\vee\vee}) = 1$ and $c_3(E^{\vee\vee}) = 1$ and Q_E is supported on a line, possibly with embedded points. In any case, Q_E fit in an exact sequence of the form:

$$0 \to Z_E \to Q_E \to i_* \mathcal{O}_l(r) \to 0. \tag{3.25}$$

Where Z_E is the maximal 0-dimensional subsheaf of Q_E of length s, and $\mathcal{O}_l$ is the structure sheaf of a line $i : l \hookrightarrow \mathbb{P}^3$. Then we have that the Euler characteristic of $Q_E(t)$ is :

$$\chi(Q_E(t)) = t + r + s + 1 \tag{3.26}$$

From sequence 3.24, and 3.26 we have that:

$$-1 = \chi(E) = \chi(E^{\vee\vee}) - \chi(Q_E) = -2 - r - s \tag{3.27}$$

hence $-r-1 = 1$, and since we have an epimorphism $E^{\vee\vee} \to Q_E$, from the equation (3.16), we have that $r \geqslant -1$, then it implies that the only possible values for r and s are $r = -1$ and $s = 0$. From what we have that $Q_E \simeq i_*\mathcal{O}_l(-1)$, additionally $c_2(E^{\vee\vee}) = c_3(E^{\vee\vee}) = 1$. Therefore, if $l \cap \operatorname{Sing}(E^{\vee\vee}) = \varnothing$, then E belongs to $\mathcal{X}(1, 1, -1, 0)$ which has dimension 5 which is too small to fill out an irreducible component of $\mathcal{M}(-1, 2, 4)$. If $l \cap \operatorname{Sing}(E) \neq \varnothing$, then by definition $E \in Y(-1)$ that has dimension 5 by Lemma 100, and also is too small to fulfill an irreducible component of $\mathcal{M}(-1, 2, 4)$.

iii) If $\operatorname{mult} Q_E = 0$, then $c_3(E^{\vee\vee}) = c_3(E) + 2s$, with $s = \operatorname{lenght}(Q_E)$, and $s > 0$ because E is not reflexive by assumption, the by Proposition 99 c), $c_3(E^{\vee\vee} > c_3(E)$, but this contradicts the stability of $E^{\vee\vee}$ since $c_2(E) = c_2(E^{\vee\vee})$.

In conclusion, we have proved that $\mathcal{M}(-1, 2, 4) = \overline{\mathcal{R}(-1, 2, 4)}$. □

Next, we will study the moduli space $\mathcal{M}(-1, 2, 2)$.

Theorem 102. The moduli space of rank 2 stable sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = -1, c_2 = 2, c_3 = 2$, has at least 2 irreducible components, namely:

a) The closure $\overline{\mathcal{R}(-1, 2, 2)}$ of the family of reflexive sheaves $\mathcal{R}(-1, 2, 2)$, of dimension 11;

b) The irreducible component $\mathrm{T}(-1, 2, 4, 1)$ given by the Theorem 96, of dimension 15, whose generic element is a torsion free sheaf E such that $E^{\vee\vee} \in \mathcal{R}(-1, 2, 4)$ and Q_E is supported in a point.

Furthemore, the intersection of these two components is non-empty.

Proof. By [11, Thm 2.5], $\mathcal{R}(-1,2,2)$ is irreducible, nonsingular of dimension 11, and its closure in $\mathcal{M}(-1,2,2)$ in $\overline{\mathcal{R}(-1,2,2)}$ is an irreducible component of $\mathcal{M}(-1,2,2)$ of dimension 11. By Theorem 96, $\mathrm{T}(-1,2,4,1)$ is an irreducible component of $\mathcal{M}(-1,2,2)$ with dimension 15. To see that these two families have non-empty intersection, consider a 1-dimensional flat family of curves $\mathcal{Z}$ in $\mathbb{P}^3$, such that :

$$\pi : \mathcal{Z} \to \mathbb{P}^3 \times U \xrightarrow{pr_2} U$$

with base U open dense of $\mathbb{A}^1$, such that $0 \in U$, satisfying:

a) for $t \neq 0$, the fiber $Z_t := \pi^{-1}(t)$ is the disjoint union of two skew lines;

b) the fiber at 0 is the union of two lines meeting in a fat point of multiplicity 2. That is, $(Z_0)_{red} = l_1 \cup l_2$ and $p = l_1 \cap l_2$, and as scheme Z_0 has an embedded point p:

$$0 \to \mathcal{O}_p \to \mathcal{O}_{Z_0} \to \mathcal{O}_{(Z_0)_{red}} \to 0 \tag{3.28}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_U \to \boldsymbol{G} \to I_{\mathcal{Z}/\mathbb{P}^3 \times U} \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_U \to 0 \tag{3.29}$$

Set $G_t := \boldsymbol{G}|_{\mathbb{P}^3} \times \{t\}$, with $t \in U$. Note that, for $t \neq 0$, each $G_t \in \mathcal{R}(-1,2,2)$ by [11, Lemma 2.4], and that for G_0 we have the following exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{r} G_0 \to I_{Z_0/\mathbb{P}^3} \to 0 \tag{3.30}$$

From the sequences (3.28) and (3.30), we have the following exact sequences:

$$0 \to G_0 \to G_0^{\vee\vee} \to \mathcal{O}_p \to 0, \tag{3.31}$$

$$0 \to \mathcal{O}(-1) \xrightarrow{s} G_0^{\vee\vee} \to I_{(Z_0)_{red}/\mathbb{P}^3} \to 0. \tag{3.32}$$

where s is the composition morphism r in the sequence (3.30) with the standard monomorphism $G_0 \to G_0^{\vee\vee}$. From sequence (3.32) and [25, Proposition 4.2] we conclude that $G_0^{\vee\vee}$ is stable, hence so is G_0.

Thus, we have a morphism :

$$\Phi_U :\ U \to \mathcal{M}(-1,2,2),\ t \mapsto [G_t], \quad G_t := \boldsymbol{G}|_{\mathbb{P}^3 \times \{t\}}. \tag{3.33}$$

Since $\Phi_U(U\backslash\{0\}) \subset \mathcal{R}(-1,2,2)$, we have that $G_0 \in \overline{\mathcal{R}(-1,2,2)}$. Moreover, by sequence (3.31), $G_0 \in \mathrm{T}(-1,2,4,1)$, which yields the proof since, by Theorem 102, we have that $\mathcal{M}(-1,2,2) = \overline{\mathcal{R}(-1,2,2)} \cup \mathrm{X}(-1,2,4,1)$. □

We note that is possible to prove that the components appearing the previous Theorem are all the possible irreducible components of $\mathcal{M}(-1,2,2)$. This can be proved by studying all the possible deformations of the sheaves in $\mathcal{M}(-1,2,2)$, and a proof can be found in [1, Theorem 24].

Next, we study the irreducible components of $\mathcal{M}(-1,2,0)$.

Theorem 103. The moduli space of rank 2 stable sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = -1, c_2 = 2, c_3 = 0$, $\mathcal{M}(-1,2,0)$, has at least 4 irreducible components, namely:

a) The closure of the family of stable rank 2 locally free sheaves which can be obtained Serre's construction as extensions of ideal sheaves of two irreducible conics (see [23, Example 9.1.2]), denoted by $\mathcal{C}(2)$, which is smooth, of dimension 11;

b) The irreducible component $\mathrm{X}(-1,1,1,1,0)$ of dimension 11, described by Theorem 3.19, whose generic element is a torsion free sheaf E such that $E^{\vee\vee} \in \mathcal{R}(-1,1,1)$ and Q_E is supported on a line.

c) The irreducible component $\mathrm{T}(-1,2,2,1)$ of dimension 15 described in Theorem 96, whose generic sheaf is a torsion free sheaf E such that $E^{\vee\vee} \in \mathcal{R}(-1,2,2)$ and Q_E is a length 1 sheaf, supported at a point.

d) The irreducible component $\mathrm{T}(-1,2,4,2)$ of dimension 19 described by the Theorem 96, whose generic sheaf is a torsion free sheaf E such that $E^{\vee\vee} \in \mathcal{R}(-1,2,4)$ and Q_E is a length 2 sheaf, supported at two distinct points.

Moreover, the union $\overline{\mathcal{C}(2)} \cup \mathrm{T}(-1,2,4,2) \cup \mathrm{T}(-1,2,2,1)$ is connected.

Proof. By [23, Proposition 4.1], $\overline{\mathcal{C}(2)}$ is an irreducible component of $\mathcal{M}(-1,2,0)$ of dimension 11. By Theorem 96, $\mathrm{T}(-1,2,2,1)$ and $\mathrm{T}(-1,2,4,2)$ are irreducible components of $\mathcal{M}(-1,2,0)$ of dimension 15 and 19, respectively. By Theorem 95, $\mathrm{X}(-1,1,1,1,0)$ is an irreducible component of $\mathcal{M}(-1,2,0)$ of dimension 11.

To see that the union $\overline{\mathcal{C}(2)} \cup \mathrm{T}(-1,2,4,2) \cup \mathrm{T}(-1,2,2,1)$ is connected, we are going to prove that the component $\overline{\mathcal{C}(2)}$ intersects $\mathrm{T}(-1,2,4,2)$ and $\mathrm{T}(-1,2,2,1)$.

Consider the following two, 1-dimensional flat families of curves $\mathcal{Z}^1$, $\mathcal{Z}^2$ in $\mathbb{P}^3$, such that for each $i = 1,2$, the family $\mathcal{Z}^i$ satisfies the conditions a) and b_i):

$$\pi : \mathcal{Z}^i \to \mathbb{P}^3 \times U \xrightarrow{pr_2} U,$$

with base U open dense of $\mathbb{A}^1$, such that $0 \in U$, satisfying:

a) for $t \neq 0$, the fiber $Z_t := \pi^{-1}(t)$ is the disjoint union of two conics;

b1) the fiber at 0 is the union of two conics, C_1 and C_2 meeting in a fat point of multiplicity 2. That is, $(Z_0^1)_{red} = C_1 \cup C_2$ and $p = C_1 \cap C_2$, and as scheme Z_0^1 has an embedded point p:

$$0 \to \mathcal{O}_p \to \mathcal{O}_{Z_0^1} \to \mathcal{O}_{(Z_0^1)_{red}} \to 0 \tag{3.34}$$

b2) the fiber at 0 is the union of two conics, C_1 and C_2 meeting in two distinct fat points of multiplicity 2. That is, $(Z_0^2)_{red} = C_1 \cup C_2$, and $\{p_1, p_2\} = C_1 \cap C_2$, and as scheme Z_0^2 has two embedded points p_1 and p_2:

$$0 \to \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \to \mathcal{O}_{Z_0^2} \to \mathcal{O}_{(Z_0^2)_{red}} \to 0 \tag{3.35}$$

Let $p_2 : \mathbb{P}^3 \times U \to U$ be the projection. For each $t \in U$, we have that $\dim \operatorname{Ext}^1(\mathcal{I}_{Z_t^i}(1), \mathcal{O}_{\mathbb{P}^3}(-2)) = 2$, and $\operatorname{Ext}^j(\mathcal{I}_{Z_t^i}, \mathcal{O}_{\mathbb{P}^3}(-2)) = 0$, for $j \geqslant 2$. Indeed, for each Z_t^i, consider the conics C_{1t}^i and C_{2t}^i, with $i = 1, 2$ such that Z_t^i fits into an exact sequence of the form:

$$0 \to \mathcal{I}_{Z_t^i} \to \mathcal{I}_{C_{1t}^i} \to \mathcal{O}_{C_{2t}^i} \to 0. \tag{3.36}$$

Moreover, each C_{jt}^i, with $i, j = 1, 2$ fits in the following exact sequences:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{I}_{C_{jt}^i} \to 0,$$

$$0 \to \mathcal{I}_{C_{jt}^i} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{C_{jt}^i} \to 0$$

twisting by $\mathcal{O}_{\mathbb{P}^3}(1)$ and applying the functor $\operatorname{Hom}(-, \mathcal{O}_{\mathbb{P}^3}(-2))$ in the above short exact sequences, we see that $\dim \operatorname{Ext}^1(\mathcal{I}_{C_{jt}^i}(1), \mathcal{O}_{\mathbb{P}^3}(-2)) = 1$ and $\dim \operatorname{Ext}^l(\mathcal{I}_{C_{jt}^i}(1), \mathcal{O}_{\mathbb{P}^3}(-2)) = 0$ if $l \neq 1$, and that $\dim \operatorname{Ext}^2(\mathcal{O}_{C_{jt}^i}(1), \mathcal{O}_{\mathbb{P}^3}(-2)) = 1$ and and $\dim \operatorname{Ext}^l(\mathcal{O}_{C_{jt}^i}(1), \mathcal{O}_{\mathbb{P}^3}(-2)) = 0$ if $l \neq 2$. Thus twisting by $\mathcal{O}_{\mathbb{P}^3}(1)$ and applying the functor $\operatorname{Hom}(-, \mathcal{O}_{\mathbb{P}^3}(-2))$ in the sequence (3.36) we get that $\dim \operatorname{Ext}^1(\mathcal{I}_{Z_t^i}(1), \mathcal{O}_{\mathbb{P}^3}(-2)) = 2$ and $\operatorname{Ext}^j(\mathcal{I}_{Z_t^i}, \mathcal{O}_{\mathbb{P}^3}(-2)) = 0$, for $j \geqslant 2$, as we claimed.

Therefore the base change for relative Ext-sheaves (see [45, Thm. 1.4]) shows that the sheaf $\mathcal{A} = \mathcal{E}xt^1_{p_2}(\mathcal{I}_{\mathcal{Z}, \mathbb{P}^3 \times \mathbb{A}^1}, \mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes \mathcal{O}_{\mathbb{A}^1})$ is a locally free $\mathcal{O}_{\mathbb{A}^1}$-sheaf and there exists a nowhere vanishing section $s \in \mathrm{H}^0(\mathcal{A})$. Furthermore, by the spectral sequence of global-to-relative Ext we may consider s as an element of the group $\operatorname{Ext}^1_{p_2}(\mathcal{I}_{\mathcal{Z}, \mathbb{P}^3 \times \mathbb{A}^1}, \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathbb{A}^1})$. Hence, this element defines an extension of $\mathcal{O}_{\mathbb{P}^3 \times U}$-sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes \mathcal{O}_U \to \mathbf{E}^i \to \mathcal{I}_{\mathcal{Z}^i, \mathbb{P}^3 \times U}(1) \boxtimes \mathcal{O}_U \to 0, \quad i = 1, 2. \tag{3.37}$$

The sheaves $\mathbf{E}^i$ are flat over U and, by construction, for $t \in U$, the restriction of (3.37) is nonsplitting extension of $\mathcal{O}_{\mathbb{P}^3}$-sheaves $0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to E_t^i \to \mathcal{I}_{Z_t^i, \mathbb{P}^3}(1) \to 0$, where

$E_t^i := \mathbf{E}^i|_{\mathbb{P}^3\times\{t\}}$. Hence, $[E_t^i] \in \mathcal{M}(-1,2,0)$, i. e., we obtain modular morphisms $\Phi_i : U \to \mathcal{M}(-1,2,0)$, $t \mapsto [E_t^i]$. Note that, for $t \neq 0$, each $[E_t^i] \in \mathcal{C}(2)$ by [23, Example 3.1.2]. Hence, also $[E_0^i] \in \overline{\mathcal{C}(2)}$, $i = 1,2$. Besides, E_0^i fit in the following exact triples:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{r_i} E_0^i \to \mathcal{I}_{Z_0^i,\mathbb{P}^3}(1) \to 0, \quad i = 1,2. \tag{3.38}$$

The triples (3.34), (3.35) and (3.38), give us the following exact sequences:

$$0 \to E_0^1 \to (E_0^1)^{\vee\vee} \to \mathcal{O}_p \to 0, \tag{3.39}$$

$$0 \to E_0^2 \to (E_0^2)^{\vee\vee} \to \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \to 0, \tag{3.40}$$

$$0 \to \mathcal{O}(-2) \xrightarrow{s_i} (E_0^i)^{\vee\vee} \to \mathcal{I}_{(Z_0^i)_{red},\mathbb{P}^3}(1) \to 0, \qquad i = 1,2, \tag{3.41}$$

where s_i is the composition morphism r_i from (3.38) with the canonical monomorphism $E_0^i \to (E_0^i)^{\vee\vee}$. From sequence (3.41) and [25, Proposition 4.2] we see that $(E_0^i)^{\vee\vee}$ is stable, and computing the Chern classes we conclude that $[(E_0^i)^{\vee\vee}] \in \mathcal{M}(-1,2,2i)$, $i = 1,2$. Thus, it follows from (3.39) that $[E_0^1] \in \mathrm{T}(-1,2,2,1)$; and, from (3.40) that $[E_0^2] \in \mathrm{T}(-1,2,4,2)$. Since, by the above, $[E_0^1], [E_0^2] \in \overline{\mathcal{C}(2)}$, $i = 1,2$, it follows, that $\overline{\mathcal{C}(2)} \cap \mathrm{T}(-1,2,2,1) \neq \varnothing$ and $\overline{\mathcal{C}(2)} \cap \mathrm{T}(-1,2,4,2) \neq \varnothing$, as we wanted. □

We note that is possible to prove that the components appearing the previous Theorem are all the possible irreducible components of $\mathcal{M}(-1,2,0)$. This can be proved by studying all the possible deformations of the sheaves in $\mathcal{M}(-1,2,0)$, and a proof can be found in [1, Theorem 27]. Moreover, it is also possible to prove that $\mathcal{M}(-1,2,0)$ is connected see [1, Theorem 29].

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